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安定化補償器のパラメトリゼーションにおけるパラメータの縮約について

Parameter Reduction of Stabilizing Controllers

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Abstract

A parameterization method of stabilizing controllers is presented. The result of this paper does not assume the existence of the coprime factorizability of plants. Under this condition, the number of parameters of the method is less than or equal to that of the previous one. Further the result is a unification of the previous parameterization methods. This paper employs neither the localization nor the local-global principle which need the knowledge of abstract algebras.

1 Introduction

Once existence of the doubly coprime factorizations comes to be known, it is easy to parameterize all stabilizing controllers by the Youla-Kučera parameterization [3, 15, 19, 20, 21, 4]. On the other hand, for some models of control systems, it is not known yet whether or not a stabilizable plant always has its doubly coprime factorization. The multidimensional systems with structural stability is one of such models [7, 8]. Further it is known that there are models such that some stabilizable plants do not have coprime factorizations [1, 10]. In the case of neutral systems for fractional exponential delay systems, methods to find doubly coprime factorizations are still under study [2]. In order to parameterize stabilizing controllers of such models, the author has recently presented a parameterization method that can be applied even to the stabilizable plant that has no doubly coprime factorization [12]. However the method needs a large number of parameters compared to the Youla-Kučera parameterization. In fact, letting m and n be the numbers of inputs and outputs, respectively, the Youla-Kučera parameterization requires mn parameters and, on the other hand, the parameterization of [12] requires $(m+n)^2$ parameters.

The objective of this paper is to present an alternative parameterization method of stabilizing controllers without the coprime factorizability in which the number of parameters is less than or equal to that of the previous method [12]. The result obtained in this paper is a unification of the Youla-Kučera parameterization and the methods given in [12, 13]. Further this paper requires neither the localization nor the local-global principle [5] such as in [12, 17, 14]. Thus the deep knowledge of abstract algebras including the module theory is not required.

The first key of our approach to the parameterization is to use a transfer matrix F such that matrix $\mathrm{Diag}(F, P)$ has a doubly coprime factorization, where P denotes a plant we want to stabilize and Diag a block diagonal matrix. It is known such a transfer matrix F exists ([20, Theorem 4.1, p.889] and Lemma 2). One more key is that by using such a matrix $\mathrm{Diag}(F, P)$, we find a stabilizing controller of P (Theorem 1). When the size of F is $m' \times n'$, the parameterization of this paper requires $(m+n')(m'+n)$ parameters. Here m' and n' can be less than or equal to m and n, respectively (Lemma 2). As a result, the number of parameters depends on the transfer matrix F and varies between mn and $(m+n)^2$. The minimal number of parameters is same as the case of the Youla-Kučera parameterization, which is the case when there exists a doubly coprime factorization (Section 6.1). On the other hand, the maximal number is same as the case of the parameterization of [12] (Section 6.2).

The approach we use in this paper is the coordinate-free approach [17, 16, 18]. The coordinate-free approach is a factorization approach [3, 15, 19, 20] without the coprime factorizability.

The paper is organized as follows. In Section 2, we introduce the coordinate-free approach. In Section 3, we give the idea of new parameterization and state main two results of this paper. They are proved in Sections 4 and 5, which also include intermediate results.

In Section 6, we compare the result of this paper with the previous parameterization methods. Section 7 gives examples of parameterizations. Conclusions are drawn in Section 8.

2 Coordinate-Free Approach

In this section we briefly review the coordinate-free approach. Sule in [17, 18] presented a theory of the feedback stabilization of multi-input multi-output strictly causal plants over commutative rings with some restrictions. This approach to the stabilization theory is called "coordinate-free approach" [16] in the sense that the coprime factorizability of transfer matrices is not required.

We consider that the set of stable causal transfer functions is a commutative ring, denoted by $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \mid n, d \in \mathcal{A},\ d\text{: nonzerodivisor}\}$. This $\mathcal{F}$ is considered as the set of all possible transfer functions. Matrices over $\mathcal{F}$ are transfer matrices. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$ such that $\mathcal{Z}$ includes all zerodivisors. Define the subsets $\mathcal{P}$ and $\mathcal{P}_{\mathrm{S}}$ of $\mathcal{F}$ as follows:

$$\mathcal{P} = \{a/b \in \mathcal{F} \mid a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\},$$
$$\mathcal{P}_{\mathrm{S}} = \{a/b \in \mathcal{F} \mid a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}.$$

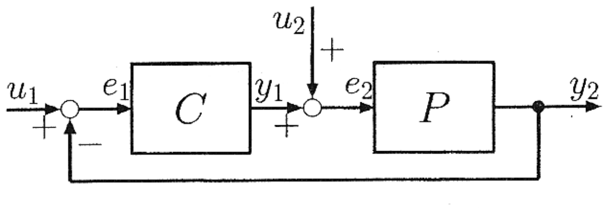

Figure 1: Feedback system Σ.

Then, every transfer function in $\mathcal{P}$ ($\mathcal{P}_S$) is called *causal* (*strictly causal*). Analogously, if every entry of a transfer matrix is in $\mathcal{P}$ ($\mathcal{P}_S$), the transfer matrix is called *causal* (*strictly causal*).

Let $\mathcal{R}$ denote $\mathcal{A}$ or $\mathcal{F}$. The set of matrices over $\mathcal{R}$ of size $x \times y$ is denoted by $\mathcal{R}^{x \times y}$. Further, the set of square matrices over $\mathcal{R}$ of size x is denoted by $(\mathcal{R})_x$. The identity and the zero matrices are denoted by E_x and $O_{x \times y}$, respectively, if the sizes are required, otherwise they are denoted by E and O. A matrix over $\mathcal{A}$ is said to be *nonsingular* if the determinant is a nonzerodivisor of $\mathcal{A}$, and *singular* otherwise. Also a matrix over $\mathcal{A}$ is said to be $\mathcal{Z}$*-nonsingular* if the determinant is in $\mathcal{A} \backslash \mathcal{Z}$, and $\mathcal{Z}$*-singular* otherwise.

The stabilization problem considered in this paper follows that of Sule in [17], and Mori and Abe in [14], who consider the feedback system Σ [19, Ch.5, Fig. 5.1] as in Fig. 1. For further details the reader is referred to [19, 17, 14]. Throughout the paper, the plant we consider has m inputs and n outputs, and its transfer matrix, which is also called a *plant* itself simply, is denoted by P and belongs to $\mathcal{P}^{n \times m}$. We can always represent P in the form of a fraction $P = ND^{-1}$ ($P = \widetilde{D}^{-1}\widetilde{N}$), where $N \in \mathcal{A}^{n \times m}$ ($\widetilde{N} \in \mathcal{A}^{n \times m}$) and $D \in \mathcal{A}^{m \times m}$ ($\widetilde{D} \in \mathcal{A}^{n \times n}$) with nonsingular D ($\widetilde{D}$).

For $P \in \mathcal{F}^{n \times m}$ and $C \in \mathcal{F}^{m \times n}$, a matrix $H(P, C) \in (\mathcal{F})_{m+n}$ is defined as

$$H(P,C) := \begin{bmatrix} (E_n + PC)^{-1} & -P(E_m + CP)^{-1} \\ C(E_n + PC)^{-1} & (E_m + CP)^{-1} \end{bmatrix} \quad (1)$$

provided that $\det(E_n + PC)$ is a nonzerodivisor of $\mathcal{A}$. This $H(P,C)$ is the transfer matrix from $[u_1^t \ \ u_2^t]^t$ to $[e_1^t \ \ e_2^t]^t$ of the feedback system Σ. If $\det(E_n + PC)$ is a nonzerodivisor of $\mathcal{A}$ and $H(P,C) \in (\mathcal{A})_{m+n}$, then we say that the plant P is *stabilizable*, P is *stabilized* by C, and C is a *stabilizing controller* of P. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [14].

Matrices A and B over $\mathcal{A}$ are *right-(left-)coprime* if there exist matrices X and Y over $\mathcal{A}$ such that $XA + YB = E$ ($AX + BY = E$) holds. Further, an ordered pair (N, D) of matrices N and D over $\mathcal{A}$ is said to be a *right-coprime factorization* of P if 1) D is nonsingular, 2) $P = ND^{-1}$ over $\mathcal{F}$, and 3) N and D are right-coprime. As the parallel notion, the *left-coprime factorization* of P is defined analogously. When P has both a right- and a left-coprime factorizations, P is said to have a *doubly coprime factorization*. Unlike [12, 17, 14], this paper does not use coprime factorizations over the ring of fractions of $\mathcal{A}$ with respect to $\{f^n \mid n \geq 0\}$ with $f \in \mathcal{A}$.

Similarly to the previous result of [12], our parameterization will be principally expressed by the set of $H(P,C)$'s with stabilizing controllers C. We denote by $\mathcal{H}(P)$ the set of all $H(P,C)$'s such that C is a stabilizing controller of P and by $\mathcal{S}(P)$ the set of all stabilizing controllers. Then, the set $\mathcal{H}(P)$ is given as $\{H(P,C) \mid C \in \mathcal{S}(P)\}$. Conversely, once we obtain $\mathcal{H}(P)$, it is also easy to obtain the set $\mathcal{S}(P)$ as follows:

$$\mathcal{S}(P) = \left\{ H_{22}^{-1} H_{21} \;\middle|\; \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathcal{H}(P) \right\} \quad (2)$$

where $H_{11} \in (\mathcal{A})_n$, $H_{12} \in \mathcal{A}^{n \times m}$, $H_{21} \in \mathcal{A}^{m \times n}$, $H_{22} \in (\mathcal{A})_m$ ([12, Lemma 2]). This implies that obtaining $\mathcal{S}(P)$ and obtaining $\mathcal{H}(P)$ are equivalent to each other.

3 Idea and Result

Here we describe the idea of the parameterization and the result of this paper. As in the previous section, we will give the set $\mathcal{H}(P)$ of $H(P,C)$'s with all stabilizing controllers C rather than the set of all stabilizing controllers itself. However our parameterization will be based on an extension of the set $\mathcal{H}(P)$. For any $H(P,C)$ in $\mathcal{H}(P)$, the determinant of $H(P,C)$ should be a nonzerodivisor. Suppose here that P and C are expressed as ND^{-1} and $\widetilde{X}^{-1}\widetilde{Y}$, respectively, where N, D, $\widetilde{Y}$, $\widetilde{X}$ are matrices over $\mathcal{A}$. Then, $H(P,C)$ is expressed as

$$\begin{bmatrix} E_n - N(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{Y} & -N(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X} \\ D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{Y} & D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X} \end{bmatrix}. \quad (3)$$

Because we will use the matrix of (3) frequently, denote it by $\widehat{H}(N, D, \widetilde{Y}, \widetilde{X})$. In order that C is a stabilizing controller, the matrix $\widetilde{X}$ needs nonsingular. On the other hand, when only considering the matrix of (3), we do not require the nonsingularity of $\widetilde{X}$. We do not want to exclude the case where $\widetilde{X}$ is singular. Thus we employ the set of all matrices in the form of (3), which still includes the set $\mathcal{H}(P)$. The new set is denoted by $\widehat{\mathcal{H}}(P)$ and defined as follows:

$$\begin{aligned} \widehat{\mathcal{H}}(P) \;=\; & \{\widehat{H}(N, D, \widetilde{Y}, \widetilde{X}) \in (\mathcal{A})_{m+n} \mid \\ & N, D, \widetilde{Y}, \widetilde{X} \text{ are matrices over } \mathcal{A}, \\ & D, \widetilde{Y}N + \widetilde{X}D \text{ are nonsingular}, \\ & P = ND^{-1} \text{ over } \mathcal{F}\}. \end{aligned} \quad (4)$$

We will characterize this $\widehat{\mathcal{H}}(P)$ by one parameter matrix. Once having the set $\widehat{\mathcal{H}}(P)$, we can obtain the parameterization of all stabilizing controllers (Theorem 2).

Suppose that the plant P is stabilizable. However we do not mention the coprime factorizability of the plant. Even so, it will be shown as Lemma 2 in the next section that there exists a transfer matrix F such that $\mathrm{Diag}(F, P)$ has a doubly coprime factorization. Denote by P_1 this $\mathrm{Diag}(F, P)$. The stabilizing controllers of P_1 can be parameterized by the Youla-Kučera parameterization [9].

For a moment, consider block diagrams in Fig. 2 as well. The new plant P_1, which is composed of P and F, can be depicted as a) of Fig. 2. Denote by C_1 a stabilizing controller of P_1. The feedback system consisting of P_1 and C_1 is as in b) of Fig. 2. Decompose the stabilizing controller C_1 as follows:

$$\begin{bmatrix} C_{111} & C_{112} \\ C_{121} & C_{122} \end{bmatrix} = C_1$$

where $C_{111} \in \mathcal{F}^{n' \times m'}$, $C_{112} \in \mathcal{F}^{n' \times n}$, $C_{121} \in \mathcal{F}^{m \times m'}$, $C_{122} \in \mathcal{F}^{m \times n}$. Consider now the following matrix

$$V_l H(P_1, C_1) V_r \quad (5)$$

where

$$V_l = \begin{bmatrix} O_{n\times m'} & E_n & O_{n\times n'} & O_{n\times m} \\ O_{m\times m'} & O_{m\times n} & O_{m\times n'} & E_m \end{bmatrix},$$

$$V_r = \begin{bmatrix} O_{m'\times n} & O_{m'\times m} \\ E_n & O_{n\times m} \\ O_{n'\times n} & O_{n'\times m} \\ O_{m\times n} & E_m \end{bmatrix}.$$

The matrix of (5) is obtained from $H(P_1, C_1)$ by letting u_{11} and u_{21} be zero and by veiling e_{11} and e_{21}. As in c) of Fig. 2, the matrix of (5) is the transfer matrix from $[\,u_{12}\ \ u_{22}\,]^t$ to $[\,e_{12}\ \ e_{22}\,]^t$. Assume here that the matrix $E_{m'} + FC_{111}$ is nonsingular. Let

$$C_{\text{New}} = C_{122} - C_{121}(E_{m'} + FC_{111})^{-1}FC_{112} \in \mathcal{F}^{m\times n}.$$

Using this C_{New}, c) of Fig. 2 can be rewritten as d) of the figure. This is a feedback system of P and C_{New}. One can check straightforwardly but tediously that the matrix of (5) is equal to $H(P, C_{\text{New}})$. The characterization of the matrix in (5) will become our first principal result of this paper, which is stated as follows.

Theorem 1 *Let P be a causal plant of $\mathcal{P}^{n\times m}$ and F a stabilizable transfer matrix of $\mathcal{F}^{m'\times n'}$. Then, we have*

$$\widehat{\mathcal{H}}(P) = \{V_l H V_r \mid H \in \widehat{\mathcal{H}}(\text{Diag}(F,P))\}. \qquad (6)$$

■

In the theorem, we have avoided to deal with the case of the matrix $E_{m'} + FC_{111}$ being singular, so that the result has become simple and intuitive. We will prove Theorem 1 in the next section.

From Theorem 1, if $\widehat{\mathcal{H}}(\text{Diag}(F,P))$ can be parameterized, then $\widehat{\mathcal{H}}(P)$ can also be parameterized. By assuming that $\text{Diag}(F,P)$ has a doubly coprime factorization, the parameterization of the set $\mathcal{H}(P)$ is stated as follows.

Theorem 2 *Let P be a causal stabilizable plant of $\mathcal{P}^{n\times m}$ and F a stabilizable transfer matrix of $\mathcal{F}^{m'\times n'}$ $(m', n' \geq 0)$ such that* $\text{Diag}(F,P)$ *has a doubly coprime factorization. Let (N_1, D_1) and $(\widetilde{D}_1, \widetilde{N}_1)$ be a right- and a left-coprime factorizations of* $\text{Diag}(F,P)$ *such that*

$$\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1 = E_{m+n'}, \quad \widetilde{N}_1 Y_1 + \widetilde{D}_1 X_1 = E_{m'+n} \qquad (7)$$

where $Y_1, \widetilde{Y}_1 \in \mathcal{A}^{(m+n')\times(m'+n)}$, $X_1 \in (\mathcal{A})_{m'+n}$, $\widetilde{X}_1 \in (\mathcal{A})_{m+n'}$. Then, we have

$$\mathcal{H}(P) = \left\{ H := V_l \begin{bmatrix} E_{m'+n} - N_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & -N_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \\ D_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & D_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \end{bmatrix} V_r \;\middle|\; R_1 \in \mathcal{A}^{(m+n')\times(m'+n)},\ H \text{ is nonsingular} \right\}. \qquad (8)$$

■

This theorem will be proved in Section 5.

In the theorem, m' and/or n' can be zero. The meaning of such cases is as follows. If $m' = n' = 0$, $\text{Diag}(F,P)$ is considered as P. If $m' = 0$ and $n' > 0$, then $\text{Diag}(F,P)$ is considered as matrix $[\ O\ P\ ]$, that is, a zero matrix is added from the left. The case $m' > 0$ and $n' = 0$ is analogous.

Once we have Theorem 2, the set $\mathcal{S}(P)$ of all stabilizing controllers can be given as (2).

4 Proof of Theorem 1

In this section and the next, we prove Theorems 1 and 2, respectively.

Before proving Theorem 1, we provide an easy lemma. In (4), matrices N and D can vary under the equality $P = ND^{-1}$. Nevertheless the following lemma gives that fixing N and D does not affect the set of (4).

Lemma 1 *Let N and D be arbitrary but fixed matrices over $\mathcal{A}$ such that $P = ND^{-1} \in \mathcal{P}^{n\times m}$. Then,*

$$\widehat{\mathcal{H}}(P) = \{\widehat{H}(N, D, \widetilde{Y}, \widetilde{X}) \in (\mathcal{A})_{m+n} \mid \widetilde{Y} \text{ and } \widetilde{X} \text{ are matrices over } \mathcal{A},\ \widetilde{Y}N + \widetilde{X}D \text{ are nonsingular}\}. \qquad (9)$$

Proof. The relation "$\supset$" is obvious. Thus we show the "$\subset$"-part of the proof only.

"$\subset$". Suppose that there exist matrices N', D', $\widetilde{Y}$, $\widetilde{X}$ over $\mathcal{A}$ such that $\widehat{H}(N', D', \widetilde{Y}, \widetilde{X}) \in \widehat{\mathcal{H}}(P)$. Then, it is easy to see that $\widehat{H}(N', D', \widetilde{Y}, \widetilde{X}) = \widehat{H}(N' \operatorname{adj}(D'), \det(D')E_m, \widetilde{Y}, \widetilde{X})$. Letting $N'' = N' \operatorname{adj}(D')$ and $d'' = \det(D')$, we have $\widehat{H}(N', D', \widetilde{Y}, \widetilde{X}) =$

$$\begin{bmatrix} E_n - N''(\widetilde{Y}N'' + \widetilde{X}d'')^{-1}\widetilde{Y} & -N''(\widetilde{Y}N'' + \widetilde{X}d'')^{-1}\widetilde{X} \\ d''(\widetilde{Y}N'' + \widetilde{X}d'')^{-1}\widetilde{Y} & d''(\widetilde{Y}N'' + \widetilde{X}d'')^{-1}\widetilde{X} \end{bmatrix}.$$

Because $ND^{-1} = N''d''^{-1}$, by replacing N'' by $ND^{-1}d''$, we further have

$$\widehat{H}(N', D', \widetilde{Y}, \widetilde{X}) = \begin{bmatrix} E_n - N(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{Y} & -N(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X} \\ D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{Y} & D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X} \end{bmatrix}.$$

Thus $\widehat{H}(N', D', \widetilde{Y}, \widetilde{X})$ is an element of the set in the right hand side of (9). □

Proof of Theorem 1. We show that the left hand side of (6) is a subset of the right ("$\subset$") and vice versa ("$\supset$").

"$\subset$". Suppose that there exist matrices N, D, $\widetilde{Y}$, $\widetilde{X}$ over $\mathcal{A}$ such that $\widehat{H}(N, D, \widetilde{Y}, \widetilde{X}) \in \widehat{\mathcal{H}}(P)$. Let C_F be a stabilizing controller of F. Further let

$$N_1 = \text{Diag}(F(E_{n'} + C_F F)^{-1}, N)$$
$$D_1 = \text{Diag}((E_{n'} + C_F F)^{-1}, D)$$
$$\widetilde{Y}_1 = \text{Diag}((E_{n'} + C_F F)^{-1}C_F, \widetilde{Y})$$
$$\widetilde{X}_1 = \text{Diag}((E_{n'} + C_F F)^{-1}, \widetilde{X})$$

which are matrices over $\mathcal{A}$. Then, $\widehat{H}(N_1, D_1, \widetilde{Y}_1, \widetilde{X}_1) \in \widehat{\mathcal{H}}(\text{Diag}(F,P))$. Further $\widehat{H}(N, D, \widetilde{Y}, \widetilde{X}) = V_l \widehat{H}(N_1, D_1, \widetilde{Y}_1, \widetilde{X}_1) V_r$.

"$\supset$". Denote $\text{Diag}(F,P)$ by P_1. Suppose that there exist matrices N_1, D_1, $\widetilde{Y}_1$, $\widetilde{X}_1$ over $\mathcal{A}$ such that $\widehat{H}(N_1, D_1, \widetilde{Y}_1, \widetilde{X}_1) \in \widehat{\mathcal{H}}(P_1)$. By Lemma 1, we assume without loss of generality that $N_1 = \text{Diag}(N_F, N)$, $D_1 = \text{Diag}(D_F, D)$, where N, D, N_F and D_F are matrices over $\mathcal{A}$ with $P = ND^{-1}$ and $F = N_F D_F^{-1}$. The matrix $V_l \widehat{H}(N_1, D_1, \widetilde{Y}_1, \widetilde{X}_1) V_r$ can be calculated as (10).

$$\begin{bmatrix} [\,O_{n\times m} \quad E_n\,](E_{m+n} - N_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{Y}_1)\begin{bmatrix} O_{m\times n} \\ E_n \end{bmatrix} & -[\,O_{n\times m} \quad E_n\,] N_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{X}_1 \begin{bmatrix} O_{n\times m} \\ E_m \end{bmatrix} \\ [\,O_{m\times n} \quad E_m\,] D_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{Y}_1 \begin{bmatrix} O_{m\times n} \\ E_n \end{bmatrix} & [\,O_{m\times n} \quad E_m\,] D_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{X}_1 \begin{bmatrix} O_{n\times m} \\ E_m \end{bmatrix} \end{bmatrix}. \tag{10}$$

Now we let

$$\widetilde{Y} := [\,O_{m\times n'} \; E_m\,] D_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{Y}_1 \begin{bmatrix} O_{m'\times n} \\ E_n \end{bmatrix} \tag{11}$$

$$\widetilde{X} := [\,O_{m\times n'} \; E_m\,] D_1(\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1)^{-1}\widetilde{X}_1 \begin{bmatrix} O_{n'\times m} \\ E_m \end{bmatrix}. \tag{12}$$

Using these $\widetilde{Y}$ and $\widetilde{X}$, we show that $\widehat{H}(N, D, \widetilde{Y}, \widetilde{N})$ is equal to (10). Because this needs simple (but tedious) calculations, we only show the equality of the (2,2)-block. The equalities of other blocks can be shown analogously.

Consider the (2,2)-block of $\widehat{H}(N, D, \widetilde{Y}, \widetilde{N})$, which is $D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X}$. This matrix should be equal to the (2,2)-block of (10), which is just equal to $\widetilde{X}$. Thus, it is sufficient to show $(D(\widetilde{Y}N + \widetilde{X}D)^{-1})^{-1}$ is the identify matrix. The matrix $(D(\widetilde{Y}N + \widetilde{X}D)^{-1})^{-1}$ is calculated as

$$\begin{bmatrix} O_{n'\times m} \\ E_m \end{bmatrix}^t (\widetilde{Y}_1 P_1 + \widetilde{X}_1)^{-1}(\widetilde{Y}_1 \begin{bmatrix} O_{m'\times m} \\ P \end{bmatrix} + \widetilde{X}_1 \begin{bmatrix} O_{n'\times m} \\ E_m \end{bmatrix}). \tag{13}$$

By noting that P_1 is equal to $\mathrm{Diag}(F, P)$, we see that (13) is the identity matrix. □

5 Proof of Theorem 2

In order to prove Theorem 2, we will provide three lemmas. Lemma 2 is to see that in the case when the plant P is stabilizable, there exists a transfer matrix F such that $\mathrm{Diag}(F, P)$ has a doubly coprime factorization. Lemma 3 will give that the Youla-Kučera parameterization can cover the set $\widehat{\mathcal{H}}(P)$ when the plant P has a doubly coprime factorization. Lemma 4 will give the relationship between the sets $\mathcal{H}(P)$ and $\widehat{\mathcal{H}}(P)$.

In the following, we first show the lemmas and then the proof of Theorem 2.

Lemma 2 *Suppose that plant $P \in \mathcal{P}^{n\times m}$ is stabilizable. Then, there exists a stabilizing transfer matrix $F \in \mathcal{F}^{m'\times n'}$ ($m', n' \geq 0$) such that plant $\mathrm{Diag}(F, P)$ has a doubly coprime factorization. Further m' and n' can be at most m and n, respectively.*

Proof. Let C be a stabilizing controller of P. Then, by [20, Theorem 4.1, p.889], $\mathrm{Diag}(C, P)$ has a doubly coprime factorization. It is known that a stabilizing controller itself is stabilizable. In this case, F is equal to C, and m' and n' are equal to m and n, respectively, from which it follows that the last statement of the lemma holds. □

By virtue of Lemma 2, m' and n' in Theorem 2 can be less than or equal to m and n, respectively. In the end of this section we will give an example that m' and n' are strictly less than m and n, respectively.

Lemma 3 *Suppose that $P \in \mathcal{P}^{n\times m}$ has a doubly coprime factorization. Let N, D, $\widetilde{N}$, $\widetilde{D}$, Y, X, $\widetilde{Y}$, $\widetilde{X}$ be arbitrary but fixed transfer matrices over $\mathcal{A}$ such that $P = ND^{-1} = \widetilde{D}^{-1}\widetilde{N}$, $\widetilde{Y}N + \widetilde{X}D = E_m$, and $\widetilde{N}Y + \widetilde{D}X = E_n$. Then,*

$$\widehat{\mathcal{H}}(P) = \tag{14}$$
$$\left\{ \begin{bmatrix} E_n - N(\widetilde{Y} + R\widetilde{D}) & -N(\widetilde{X} - R\widetilde{N}) \\ D(\widetilde{Y} + R\widetilde{D}) & D(\widetilde{X} - R\widetilde{N}) \end{bmatrix} \;\middle|\; R \in \mathcal{A}^{m\times n} \right\}.$$

Proof. The relation "⊃" is obvious. Thus we show the "⊂"-part of the proof only.

"⊂". Let $H \in \widehat{\mathcal{H}}(P)$. There exist matrices $\widetilde{Y}'$ and $\widetilde{X}'$ over $\mathcal{A}$ such that $H = \widehat{H}(N, D, \widetilde{Y}', \widetilde{X}')$ by Lemma 1.

Because H is over $\mathcal{A}$, the matrices $E_n - N(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}'$ and $D(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}'$ are also over $\mathcal{A}$, which are the (1,1)- and (2,1)-blocks of the matrix $\widehat{H}(N, D, \widetilde{Y}', \widetilde{X}')$, respectively. Then, we have

$$[\widetilde{Y} \quad \widetilde{X}] \begin{bmatrix} N(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}' \\ D(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}' \end{bmatrix} = (\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}'.$$

Thus, the matrix $(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}'$ is over $\mathcal{A}$. Analogously $(\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{X}'$ is also over $\mathcal{A}$. Now let $\widetilde{Y}'' = (\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{Y}'$ and $\widetilde{X}'' = (\widetilde{Y}'N + \widetilde{X}'D)^{-1}\widetilde{X}'$. Then, H is equal to $\widehat{H}(N, D, \widetilde{Y}'', \widetilde{X}'')$, which is expressed as

$$\begin{bmatrix} E_n - N\widetilde{Y}'' & -N\widetilde{X}'' \\ D\widetilde{Y}'' & D\widetilde{X}'' \end{bmatrix}.$$

Because $\widetilde{Y}''N + \widetilde{X}''D = E_m$ holds, by [9, Theorem 3.1] there exists an R in $\mathcal{A}^{m\times n}$ such that $\widetilde{Y}'' = Y + R\widetilde{D}$ and $\widetilde{X}'' = X - R\widetilde{N}$. Now H is expressed as the matrix in the set of (14). □

Lemma 4 *Let P be a causal plant of $\mathcal{P}^{n\times m}$. Then, the following equalities hold:*

$$\mathcal{H}(P) = \{H \in \widehat{\mathcal{H}}(P) \mid H \text{ is nonsingular}\} \quad (15)$$

$$\mathcal{H}(P) = \{[\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}] \in \widehat{\mathcal{H}}(P) \mid H_{11} \text{ is nonsingular}\} \quad (16)$$

$$\mathcal{H}(P) = \{[\begin{smallmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{smallmatrix}] \in \widehat{\mathcal{H}}(P) \mid H_{22} \text{ is nonsingular}\} \quad (17)$$

where $H_{11} \in (\mathcal{A})_n$, $H_{12} \in \mathcal{A}^{n\times m}$, $H_{21} \in \mathcal{A}^{m\times n}$, $H_{22} \in (\mathcal{A})_m$.

Proof. Once we obtain (17), then (15) and (16) are obvious. Thus we show (17) only.

"$\subset$". Let $H \in \mathcal{H}(P)$ and C a stabilizing controller of P with $H(P,C) = H$. Let $N = P(E_m+CP)^{-1}$, $D = (E_m+CP)^{-1}$, $\widetilde{Y} = C(E_n + PC)^{-1}$, $\widetilde{X} = (E_m + CP)^{-1}$. Then, ND^{-1} is equal to P. Further $\widetilde{Y}N + \widetilde{X}D$ is equal to $(E_m + CP)^{-1}$, which is nonsingular. Calculating $\widehat{H}(N, D, \widetilde{Y}, \widetilde{X})$, we have $H = \widehat{H}(N, D, \widetilde{Y}, \widetilde{X})$. Hence $H \in \widehat{\mathcal{H}}(P)$.

"$\supset$". Suppose $\widehat{H}(N, D, \widetilde{Y}, \widetilde{X}) \in \widehat{\mathcal{H}}(P)$. By the condition in the set of (17), the matrix $D(\widetilde{Y}N + \widetilde{X}D)^{-1}\widetilde{X}$ is nonsingular. Thus $\widetilde{X}$ is also nonsingular. By letting $C = \widetilde{X}^{-1}\widetilde{Y}$, we have $H(P, C) = \widehat{H}(N, D, \widetilde{Y}, \widetilde{X})$. □

Now we can prove Theorem 2.

Proof of Theorem 2. The existence of the transfer matrix F is from Lemma 2. By Theorem 1 and Lemma 3, the set $\widehat{H}(P)$ can be expressed as

$$\widehat{\mathcal{H}}(P) = \left\{ V_l \begin{bmatrix} E_{m'+n} - N_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & -N_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \\ D_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & D_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \end{bmatrix} V_r \;\middle|\; R_1 \in \mathcal{A}^{(m+n')\times(m'+n)} \right\}. \quad (18)$$

Combining (15) of Lemma 4 and (18), we have (8). □

Example 1 Before finishing this section, it is worth mentioning the case where m' and n' of Theorem 2 are strictly less than m and n, respectively.

Let us consider [14, Example 3.4]. In the example, $\mathcal{A}$ is equal to $\mathbb{R}[z^2, z^3]$, where z denotes a unit delay operator. The impulse response of a transfer function being stable is finitely terminated and does not have the unit delay.

Let $p = (1 - z^3)/(1 - z^2)$. This p does not have a coprime factorization but stabilizable. A stabilizing controller c of p is given as

$$c = \frac{-101 + 255z^2 - 343z^3 - 56z^4 + 343z^5 - 98z^6}{1089 - 154z^2 + 242z^3 - 98z^4 + 154z^5 - 343z^6 + 98z^7}.$$

Consider a new plant $P = \mathrm{diag}(p, 1)$. This P does not have a doubly coprime factorization as explained below. Let $N = \mathrm{diag}(1 - z^3, 1 - z^2)$ and $D = (1 - z^2)E_2$. Then, $P = ND^{-1}$ holds. In order that P has a doubly coprime factorization, the ideal generated by the full-size minors of the matrix $[\, N^t \; D^t \,]^t$ needs to be principal. The common factor of $1 - z^3$ and $1 - z^2$ over $\mathbb{R}[z]$ is $1 - z$. Any generator of the ideal has a factor $1 - z$ over $\mathbb{R}[z]$. However, $1 - z$ is not in $\mathcal{A}$. Thus, the ideal is not principle. Hence P does not have a doubly coprime factorization. On the other hand, $\mathrm{Diag}(c, P)$ has a doubly coprime factorization, because $\mathrm{diag}(c, p)$ has. In fact, letting

$$\begin{aligned}
N_1 &= \widetilde{N}_1 = \mathrm{Diag}\left(\begin{bmatrix} c(1+pc)^{-1} & -pc(1+pc)^{-1} \\ pc(1+pc)^{-1} & p(1+pc)^{-1} \end{bmatrix}, 1\right) \\
D_1 &= \mathrm{Diag}\left(\begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix}, 1\right) \\
\widetilde{D}_1 &= \mathrm{Diag}\left(\begin{bmatrix} (1+pc)^{-1} & -c(1+pc)^{-1} \\ p(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix}, 1\right) \\
Y_1 &= \widetilde{Y}_1 = \mathrm{Diag}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, 0\right) \\
X_1 &= \widetilde{X}_1 = E_3,
\end{aligned}$$

we have that (N_1, D_1) and $(\widetilde{N}_1, \widetilde{D}_1)$ are a right- and a left-coprime factorizations of $\mathrm{Diag}(c, P)$, respectively, such that (7) holds. Therefore this is the case where m' and n' $(= 1)$ are less than m and n $(= 2)$, respectively. ▪

Theorem 2 gives a parameterization method of stabilizing controllers. It is based on the parameterization of $\widehat{\mathcal{H}}(P)$. Because there is a condition "H is nonsingular" in the set of (8), we may have the possibility that even if the set $\widehat{\mathcal{H}}(P)$ is not empty, the set $\mathcal{H}(P)$ becomes empty, that is, the plant is not stabilizable. If this possibility would be affirmative, the attractiveness of Theorem 2 would be decreased. Fortunately the following proposition answers that there is no such possibility by showing that there exists a causal stabilizing controller of the plant whenever the set $\widehat{\mathcal{H}}(P)$ is not empty.

Proposition 1 *Plant $P \in \mathcal{P}^{n\times m}$ is stabilizable if and only if the set $\widehat{\mathcal{H}}(P)$ is not empty. If the set $\widehat{\mathcal{H}}(P)$ is not empty, then there exists a causal stabilizing controller of P.* ▪

This proof can be found in [11]

6 Comparison with the previous results

In this section, we compare the result of this paper with the previous results, that is, the Youla-Kučera parameterization, the parameterizations in [12] and [13]. We will see that the result of this paper is a unification of them.

6.1 Youla-Kučera Parameterization

Consider Theorem 2 and suppose that P has a doubly coprime factorization. Then, the size $m' \times n'$ of F can be 0×0, that is, $\mathrm{Diag}(F, P)$ is equal to P itself. The matrices V_l and V_r are equal to the identity matrices. Hence (8) becomes equivalent to the Youla-Kučera parameterization.

6.2 Parameterization of [12]

Let us quickly review the parameterization of [12]. Suppose that $P \in \mathcal{P}^{n\times m}$ is stabilizable. Let C be a stabilizing controller of P. Let $\Omega(Q)$ be a matrix defined as follows:

$$\Omega(Q) = \left(H(P,C) - \begin{bmatrix} E_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m} \end{bmatrix}\right) Q \left(H(P,C) - \begin{bmatrix} O_{n\times n} & O_{n\times m} \\ O_{m\times n} & E_m \end{bmatrix}\right) + H(P,C)$$

with $Q \in (\mathcal{A})_{m+n}$. Then, we have the identity

$$\mathcal{H}(P) = \{\Omega(Q) \mid Q \in (\mathcal{A})_{m+n} \text{ and } \Omega(Q) \text{ is nonsingular}\}$$

[12, Theorems 4.2 and 4.3].

The parameterization above is given by a parameter matrix Q without the coprime factorizability of the plant. Nevertheless, this always requires $(m+n)^2$ parameters.

In the following we show that the parameterization of this paper involves that of [12].

As in the proof of Lemma 2, the transfer matrix F of Theorem 2 can be the stabilizing controller C. Let

$$\begin{aligned} N_1 &= \widetilde{N}_1 = \begin{bmatrix} C(E_n+PC)^{-1} & -CP(E_m+CP)^{-1} \\ PC(E_n+PC)^{-1} & P(E_m+CP)^{-1} \end{bmatrix}, \\ D_1 &= \begin{bmatrix} (E_n+PC)^{-1} & -P(E_m+CP)^{-1} \\ C(E_n+PC)^{-1} & (E_m+CP)^{-1} \end{bmatrix}, \\ \widetilde{D}_1 &= \begin{bmatrix} (E_m+CP)^{-1} & -C(E_n+PC)^{-1} \\ P(E_m+CP)^{-1} & (E_n+PC)^{-1} \end{bmatrix}, \\ Y_1 &= \widetilde{Y}_1 = \begin{bmatrix} O & E_n \\ -E_m & O \end{bmatrix}, \quad X_1 = \widetilde{X}_1 = E_{m+n}. \end{aligned}$$

Note here that the matrices above are over $\mathcal{A}$. Hence (N_1, D_1) and $(\widetilde{D}_1, \widetilde{N}_1)$ are right- and left-coprime factorizations of $\mathrm{Diag}(C, P)$, respectively, with $\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1 = E_{m+n}$, $\widetilde{N}_1 Y_1 + \widetilde{D}_1 X_1 = E_{m+n}$. In this case, m' and n' are equal to m and n, respectively.

Consider the matrix in (8):

$$V_l \begin{bmatrix} E_{m'+n} - N_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & -N_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \\ D_1(\widetilde{Y}_1 + R_1\widetilde{D}_1) & D_1(\widetilde{X}_1 - R_1\widetilde{N}_1) \end{bmatrix} V_r. \tag{19}$$

When $R_1 = O$, the matrix of (19) is equal to $H(P,C)$. Thus, (19) can be rewritten as follows:

$$H(P,C) + V_l \begin{bmatrix} -N_1 \\ D_1 \end{bmatrix} R_1 [\,\widetilde{D}_1 \quad -\widetilde{N}_1\,] V_r.$$

Simple calculations show that

$$\begin{aligned} & V_l \begin{bmatrix} -N_1 \\ D_1 \end{bmatrix} \\ = \; & H(P,C) - \begin{bmatrix} E_n & O_{n\times m} \\ O_{m\times n} & O_{m\times m} \end{bmatrix}, \\ & \begin{bmatrix} O_{n\times m} & E_n \\ -E_m & O_{m\times n} \end{bmatrix} [\,\widetilde{D}_1 \quad -\widetilde{N}_1\,] V_r \\ = \; & H(P,C) - \begin{bmatrix} O_{n\times n} & O_{n\times m} \\ O_{m\times n} & E_m \end{bmatrix}. \end{aligned}$$

Thus $\Omega(R_1[\begin{smallmatrix} O & E_n \\ -E_m & O \end{smallmatrix}]^{-1})$ is equal to (19). Therefore the parameterization of [12] is equivalent to the parameterization of Theorem 2 with F being a stabilizing controller C.

6.3 Parameterization of [13]

Mori in [13] showed that if there exists a right-coprime factorization of plant $P \in \mathcal{P}^{n\times m}$, then the plant $[\,P^t \;\; O_{m\times m}]^t \in \mathcal{P}^{(m+n)\times m}$ has both right- and left-coprime factorizations [13, Theorem 1]. Using this result, he gave the following theorem:

Theorem 3 [13, Theorem 3] *Suppose that there exists a right-coprime factorization (N, D) over $\mathcal{A}$ of the plant $P \in \mathcal{P}^{n\times m}$ with $\widetilde{Y}N + \widetilde{X}D = E_m$. Let $(\widetilde{N}', \widetilde{D}')$ be a left-coprime factorization over $\mathcal{A}$ of the plant $[\,P^t \;\; O_{m\times m}\,]^t \in \mathcal{P}^{(m+n)\times m}$ with $\widetilde{N}'Y' + \widetilde{D}'X' = E_{m+n}$. Then, all stabilizing controllers of the plant P are of the form*

$$(\widetilde{X} - R\widetilde{N}')^{-1}([\,\widetilde{Y} \;\; O_{m\times m}\,] + R\widetilde{D}') \begin{bmatrix} E_n \\ O_{m\times n} \end{bmatrix} \tag{20}$$

with $\widetilde{X} - R\widetilde{N}'$ nonsingular, where R is a parameter matrix of $\mathcal{A}^{m\times(m+n)}$. ∎

The discussion that our parameterization can obtain the matrix expression of (20) can be found in [11].

7 Example

Here we give two examples. First one is continued from Example 1. The other is the multidimensional system with structural stability.

Example 2 (Continued from Example 1) Consider to parameterize all stabilizing controllers of the plant $\mathrm{diag}(1, p)$. Then, F of Theorem 2 is c, the stabilizing controller of p. Letting N_1, D_1, $\widetilde{N}_1$, $\widetilde{D}_1$, $\widetilde{Y}_1$, $\widetilde{X}_1$ be as in Example 1, we can obtain the set $\mathcal{H}(P)$ of (8) in Theorem 2 and so the controller parameterization.

Because $m' = n' = 1$ and $m = n = 2$ as in Example 1, the number of parameters is nine. ∎

Example 3 **The multidimensional systems with structural stability** is a model in which we do not know yet whether or not a stabilizable plant always has its doubly coprime factorization [7, 8]. This implies that we do not have a general method to obtain a doubly coprime factorization even if it exists.

In the multidimensional system with structural stability, the set of stable causal transfer functions is defined as $\mathcal{A} = \{a/b \,|\, a, b \in \mathbb{R}[z_1, \ldots, z_l],\ b \neq 0 \text{ in } \overline{U}^l\}$, where $\overline{U}^l$ denotes the closed unit polydisc. To define the causality, $\mathcal{Z}$ is defined as

$$\mathcal{Z} = \sum_{i=1}^{l} z_i \mathcal{A} = \{a/b \in \mathcal{A} \,|\, a, b \in \mathbb{R}[z_1, \ldots, z_l], \text{ the constant term of } a \text{ is zero.}\}$$

which is a prime ideal of $\mathcal{A}$.

In order to make this paper concise, let us consider the plant $P(z_1, z_2, z_3)$ in [6, (55)] and its stabilizing controller $C(z_1, z_2, z_3)$ in [6, (57)] without considering the coprime factorizability of the plant P. Denote $P(z_1, z_2, z_3)$ by P and $C(z_1, z_2, z_3)$ by C.

Consider again a new plant $P' = \text{diag}(P, 1)$. Then, $\text{Diag}(C, P')$ has a right- and a left-coprime factorizations (N_1, D_1) and $(\widetilde{N}_1, \widetilde{D}_1)$ with $\widetilde{Y}_1 N_1 + \widetilde{X}_1 D_1 = E_5$ and $\widetilde{N}_1 Y_1 + \widetilde{D}_1 X_1 = E_5$, where

$$\begin{aligned} N_1 &= \widetilde{N}_1 \\ &= \text{Diag}\left(\begin{bmatrix} C(E_2+PC)^{-1} & -CP(E_2+CP)^{-1} \\ PC(E_2+PC)^{-1} & P(E_2+CP)^{-1} \end{bmatrix}, 1\right) \\ D_1 &= \text{Diag}\left(\begin{bmatrix} (E_2+PC)^{-1} & -P(E_2+CP)^{-1} \\ C(E_2+PC)^{-1} & (E_2+CP)^{-1} \end{bmatrix}, 1\right) \\ \widetilde{D}_1 &= \text{Diag}\left(\begin{bmatrix} (E_2+CP)^{-1} & -C(E_2+PC)^{-1} \\ P(E_2+CP)^{-1} & (E_2+PC)^{-1} \end{bmatrix}, 1\right) \\ Y_1 &= \widetilde{Y}_1 = \text{Diag}\left(\begin{bmatrix} O & E_2 \\ -E_2 & O \end{bmatrix}, 0\right) \\ X_1 &= \widetilde{X}_1 = E_5. \end{aligned}$$

Finally, since $m' = n' = 2$ and $m = n = 3$, the number of parameters is 25. On the other hand, the method of [12] requires 36 parameters. ∎

8 Conclusion

In this paper, we have given a parameterization method without the coprime factorizability. The method has been obtained without abstract algebras. Further the method is a unification of the previous controller parameterization methods.

This paper has shown that the number of parameters can be reduced. In order to make the parameterization more effective, we need to find smaller-sized F (of Theorem 2). However, this paper does not give the method to find the transfer matrix F such that $\text{Diag}(F, P)$ has a doubly coprime factorization. This will depend on the plant itself as well as the set $\mathcal{A}$ of stable causal transfer functions.

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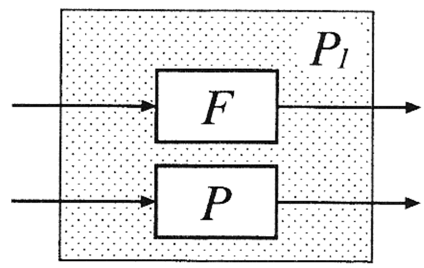
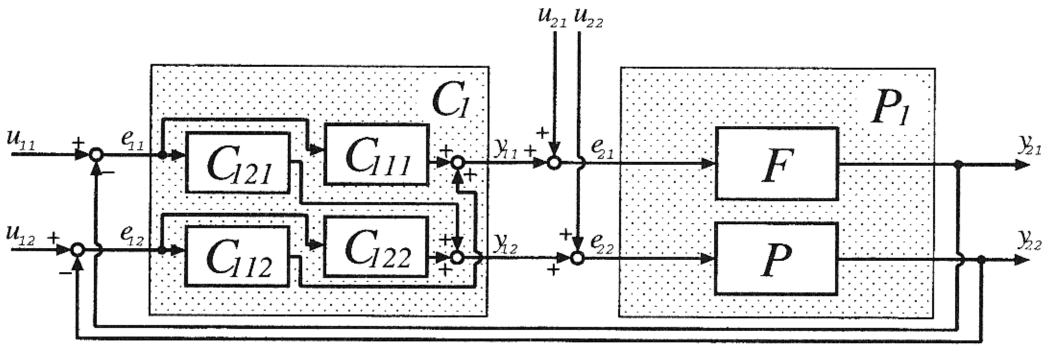
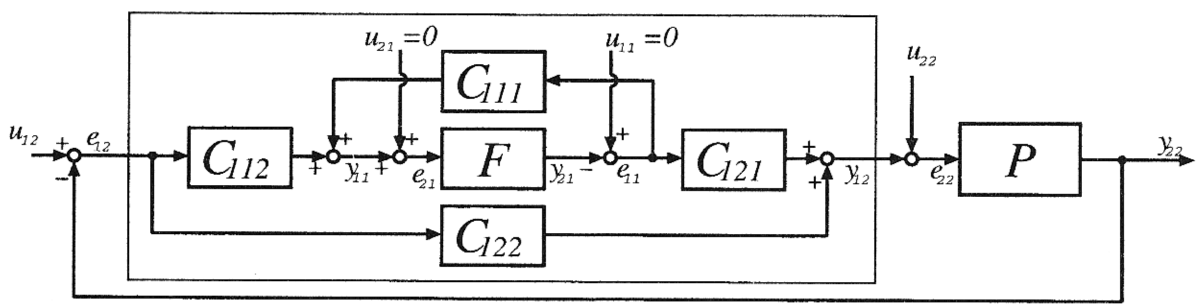


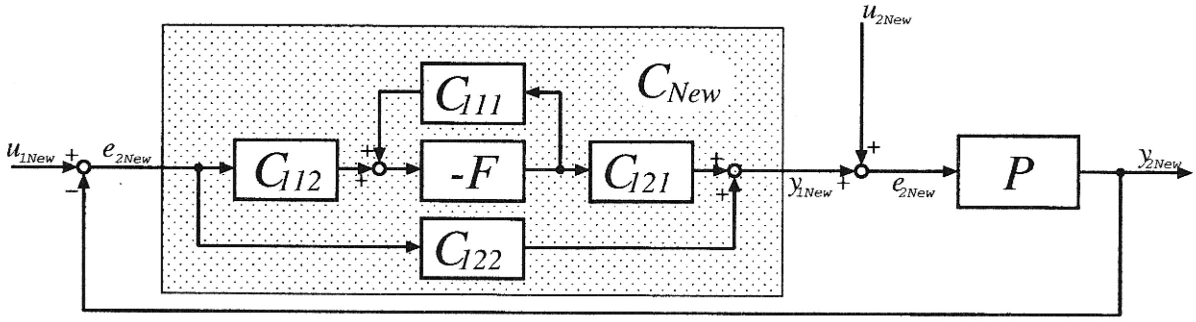
a) New plant $P_1 := \mathrm{Diag}(F, P)$.



b) New plant P_1 and its stabilizing controller C_1.



c) Relocating the components of P_1 and C_1.



d) Original plant P and its newly obtained stabilizing controller C_{New}.

Figure 2: Construction of a stabilizing controller of the plant.