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標準問題とモデルマッチング問題の等価性の一般化について

On the Relationship between Standard Problem and Model-Matching Problem

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Abstract The equivalence between the standard control problem and the model-matching problem without coprime factorizability of plants is presented.

1 Introduction

The objective of this paper is to show the equivalence between the standard control problem and the model-matching problem without the help of coprime factorization.

In the classical case, that is, in the case where there exists a doubly coprime factorization for every stabilizable plant, it is known that each problem can be recast as the other problem [F87].

On the other hand, for some models of control systems, it is not known yet whether or not a stabilizable plant always has its doubly coprime factorization. The multidimensional systems with structural stability is one of such models [Lin01, Lin99]. Further it is known that there are models such that some stabilizable plants do not have coprime factorizations [A85]. In the case of neutral systems for fractional exponential delay systems, methods to find doubly coprime factorizations are still under study [BP01].

Under these circumstances, Ball and Malakorn in [BM02] have recently stated "the reduction to the model-matching form is not obvious since the notion of coprime factorization splits in several independent ways in the nD case." This paper gives its answer and presents that the standard control problem can be reduced to a model-matching problem even if we do not consider the coprime factorizability (see Theorem 2.1 and Section 5).

2 Preliminaries

The approach we use in this note is the coordinate-free approach [S94, SS92, MA01, M02b, M03]. The coordinate-free approach is a factorization approach [DLMS80, VSF82] without coprime factorizability. We consider that the set of stable causal transfer functions is a commutative ring $\mathcal{A}$. The total ring of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is, $\mathcal{F} = \{n/d \,|\, n, d \in \mathcal{A},\ d\text{: nonzerodivisor}\}$. This is considered as the set of all possible transfer functions. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$ such that $\mathcal{Z}$ includes all zerodivisors. Define the subsets $\mathcal{P}$ and $\mathcal{P}_S$ of $\mathcal{F}$ as follows:

$$\mathcal{P} = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{A},\ b \in \mathcal{A}\backslash\mathcal{Z}\},$$
$$\mathcal{P}_S = \{a/b \in \mathcal{F} \,|\, a \in \mathcal{Z},\ b \in \mathcal{A}\backslash\mathcal{Z}\}.$$

Then, every transfer function in $\mathcal{P}$ ($\mathcal{P}_S$) is called *causal* (*strictly causal*). Analogously, if every entry of a transfer matrix is in $\mathcal{P}$ ($\mathcal{P}_S$), the transfer matrix is called *causal* (*strictly causal*).

The identity and the zero matrices are denoted by I_x and $O_{x\times y}$, respectively, if the sizes are required, otherwise they are denoted by I and O. A matrix is said to be *nonsingular* if its determinant is a nonzerodivisor, and *singular* otherwise.

We consider two types of feedback systems.

The first one is shown in Fig. 1 (cf.[DFT92, Fig. 1.2]). The transfer matrices G and K over $\mathcal{F}$ represent a generalized plant and a controller, respectively. To make the description precise, we let m and m' denote the number of inputs of G for u and w, respectively, and let n and n' the number of outputs of G for y and z, respectively. The sizes of G and K are $(n+n')\times(m+m')$ and $m \times n$, respectively. Decompose G into four blocks as follows:

$$G = \begin{matrix} & \begin{matrix} m' & m \end{matrix} \\ \begin{matrix} n' \\ n \end{matrix} & \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \end{matrix}. \qquad (1)$$

The input-output relations are:

$$\begin{bmatrix} G_{11}+G_{12}K(I-G_{22}K)^{-1}G_{21} & G_{12}(I-KG_{22})^{-1} & G_{12}K(I-G_{22}K)^{-1} \\ K(I-G_{22}K)^{-1}G_{21} & (I-KG_{22})^{-1} & K(I-G_{22}K)^{-1} \\ (I-G_{22}K)^{-1}G_{21} & G_{22}(I-KG_{22})^{-1} & (I-G_{22}K)^{-1} \end{bmatrix} \tag{2}$$

$$\begin{aligned} z &= G_{11}w + G_{12}u, \\ y &= G_{21}w + G_{22}u + v_2, \\ u &= Ky + v_1. \end{aligned}$$

Then the transfer matrix from $[\, w\ v_1\ v_2\,]^t$ to $[\, z\ u\ y\,]^t$ is given as follows:

$$\Theta(G,K) = \begin{bmatrix} I_{n'} & -G_{12} & O \\ O & I_m & -K \\ O & -G_{22} & I_n \end{bmatrix}^{-1} \begin{bmatrix} G_{11} & O & O \\ O & I_m & O \\ G_{21} & O & I_n \end{bmatrix},$$

which is equal to (2), provided that $I - KG_{22}$ is nonsingular. In particular, if the transfer matrix $\Theta(G,K)$ is over $\mathcal{A}$, then we say that K *stabilizes* G and that K is *a stabilizing controller* of G. If a stabilizing controller of G exists, then G is called *stabilizable*. The other feedback system can be considered as a part of the first one and shown in Fig. 3 (cf.[DFT92, Fig. 1.3]). The transfer matrices P and K over $\mathcal{F}$ represent a plant and its controller, respectively. The plant P has m inputs and n outputs. Let $H(P,K)$ denote the transfer matrix from $[\, v_2^t \quad v_1^t\,]^t$ to $[\, y^t \quad u^t\,]^t$, which is equal to

$$\begin{bmatrix} (I+PK)^{-1} & -P(I+KP)^{-1} \\ K(I+PK)^{-1} & (I+KP)^{-1} \end{bmatrix} \tag{3}$$

provided that $I+KP$ is nonsingular. If $I+KP$ is nonsingular and $H(P,K)$ is over $\mathcal{A}$, then we say that K *stabilizes* P and that K is *a stabilizing controller* of P. Also if a stabilizing controller of P exists, then P is called *stabilizable*.

We suppose that the norm has already been defined for matrices over $\mathcal{A}$. It can be, for example, the $\mathbf{H}_\infty$-norm. Since the norm is used only in order to introduce the problem, we do not mention its concrete definition.

Now two problems are defined as follows:

Problem 2.1 (Standard Control Problem) Consider the feedback system of Fig. 1. Causal transfer matrix G is given. Assume that G is stabilizable. Find a causal transfer matrix K that minimizes the norm of the transfer matrix form w to z under the constraint that K stabilizes G.

Problem 2.2 (Model-Matching Problem) Consider the system depicted in Fig. 2. Causal transfer matrices T_1, T_2, and T_3 over $\mathcal{A}$ are given. Find a causal transfer matrix Q over $\mathcal{A}$ such that the norm of $T_1 - T_2QT_3$ is minimum.

For the further detail of the problems, the reader is referred to [F87]. It is known that the model-matching problem and other many problems such as the tracking problem and the weighted sensitivity problem can be considered as a standard problem [F87, Zam81, Kwa85, VJ84]. Also it is known that if every stabilizing controller has a doubly coprime factorization, the standard control problem can be considered as a model-matching problem[F87]. This paper will remove the requirement of the existence of the doubly coprime factorization. Accordingly the result of this paper is that the standard control problem and the model-matching problem are equivalent even if the doubly coprime factorization is not employed. This is written as a following theorem:

Theorem 2.1 *The model-matching problem can be modified as a standard control problem. Conversely the standard control problem can be modified as a model-matching problem if G_{22} is strictly causal.*

In the case $\mathcal{A} = \mathbf{RH}_\infty$, the first statement of the theorem is given in Chapter 3 and the second is in Chapter 4 of [F87]. The first statement is proved analogously to the case of $\mathcal{A} = \mathbf{RH}_\infty$. In fact, the model-matching problem can be easily recast as a standard control problem by letting

$$G := \begin{bmatrix} T_1 & T_2 \\ T_3 & O \end{bmatrix} \text{ and } K := -Q \tag{4}$$

(see [F87, p.19]). Thus we will need to show the second statement, which will be proved in Section 4.

By assuming the strict causality of G_{22}, we have that every stabilizing controller is causal (Proposition 6.2 of [MA01]) and that the closed loop is well-posed[ZDG96, p.119] for every stabilizing controller (Proposition 5 of [M02a]).

Before finishing this section, we state the relationship between $\Theta(G,K)$ and $H(-G_{22},K)$. A straightforward calculation gives the following matrix equations:

$$\begin{aligned} \Theta(G,K) &= \begin{bmatrix} O & G_{12} \\ O & I_m \\ I_n & O \end{bmatrix} H(-G_{22},K) \begin{bmatrix} G_{21} & O & I_n \\ O & I_m & O \end{bmatrix} \\ &\quad + \begin{bmatrix} G_{11} & O & O \\ O & O & O \\ O & O & O \end{bmatrix} \end{aligned} \tag{5}$$

and

$$H(-G_{22},K) = \begin{bmatrix} O & O & I_n \\ O & I_m & O \end{bmatrix} \Theta(G,K) \begin{bmatrix} O & O \\ O & I_m \\ I_n & O \end{bmatrix}. \tag{6}$$

3 Previous Result

Here we briefly outline the parameterization method of [M02a]. Let $\mathcal{H}$ be the set of $H(P,C)$'s with all stabilizing controllers C. Let H_0 be $H(P,C_0) \in \mathcal{A}^{(m+n)\times(m+n)}$, where C_0 is a stabilizing controller of P. Let $\Omega(Q)$ be a matrix defined as

$$\Omega(Q) = \left(H_0 - \begin{bmatrix} I_n & O \\ O & O \end{bmatrix}\right) Q \left(H_0 - \begin{bmatrix} O & O \\ O & I_m \end{bmatrix}\right) + H_0 \tag{7}$$

with a stable causal and square matrix Q in $\mathcal{A}^{(m+n)\times(m+n)}$. Then we have the identity

$$\mathcal{H} = \{\Omega(Q) \mid Q \text{ is stable causal and } \Omega(Q) \text{ is nonsingular}\}$$

[M02a, Theorems 4.2 and 4.3]. Then, from (3), any stabilizing controller has the form $\Omega_{21}\Omega_{22}^{-1}$, where Ω_{21} and Ω_{22} are the (2,1)- and (2,2)-blocks of $\Omega(Q)$, provided that Ω_{22} is nonsingular.

The parameterization above is given by a parameter matrix Q without coprime factorizability of the plant.

4 Proof of Theorem 2.1

In this section we present the proof of Theorem 2.1.

The first step is to give a generalization of [F87, Theorem 4.3.2]. Then we prove Theorem 2.1.

Theorem 4.1 *Let G and K be a generalized plant and its controller over $\mathcal{F}$, respectively. Decompose G as in (1). Suppose that G is stabilizable. Then K stabilizes G if and only if K stabilizes $-G_{22}$.*

Proof. (Only if) Suppose that K stabilizes G. Then $\Theta(G,K)$ is over $\mathcal{A}$. By (6), $H(-G_{22},K)$ is also over $\mathcal{A}$. Thus K also stabilizes $-G_{22}$.
(If) Because G is stabilizable, there exists a stabilizing controller of G, say K_0. As in "(Only if)" part, this K_0 is also a stabilizing controller of $-G_{22}$. Let Θ_0 and H_0 be $\Theta(G,K_0)$ and $H(-G_{22},K_0)$, respectively, both of which are over $\mathcal{A}$.

Suppose that K is a (possibly different) stabilizing controller of $-G_{22}$. Then by the result of Section 3, there exists a matrix Q over $\mathcal{A}$ such that the following holds:

$$\begin{aligned} H(-G_{22},K) &= \left(H_0 - \begin{bmatrix} I_n & O \\ O & O \end{bmatrix}\right) \\ &\quad \times Q\left(H_0 - \begin{bmatrix} O & O \\ O & I_m \end{bmatrix}\right) + H_0. \end{aligned}$$

By virtue of (5), $\Theta(G,K)$ is expressed as (8) (see the bottom of this page). A simple calculation shows that (8) is equal to (9) The matrix above is over $\mathcal{A}$ because both Q and Θ_0 are. Hence K stabilizes G. □

From the proof of Theorem 4.1, we see that every $\Theta(G,K)$ with a stabilizing controller K is given in the form of (9) with some Q of $\mathcal{A}^{(m+n)\times(m+n)}$.

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We have shown the first statement of the theorem after the description of Theorem 2.1. Thus we here show the second statement.

Suppose that K_0 stabilizes a generalized plant G and that G_{22} is strictly causal. Let Θ_0 be as in the proof of Theorem 4.1.

From the result of Theorem 4.1, every $\Theta(G,K)$ with a stabilizing controller K is given in the form of (9). Thus the standard control problem is equivalently to find the matrix Q over $\mathcal{A}$ that minimizes the norm of the matrix of (9) multiplied by the matrix $[I_{n'} \quad O \quad O]$ from the left and multiplied by the matrix $[I_{m'} \quad O \quad O]^t$ from the right. Now let

$$\begin{aligned} T_1 &:= [I_{n'} \quad O \quad O]\Theta_0 \begin{bmatrix} I_{m'} \\ O \\ O \end{bmatrix} \\ &= G_{11} + G_{12}K_0(I - G_{22}K_0)^{-1}G_{21}, \\ T_2 &:= -[I_{n'} \quad O \quad O]\left(\Theta_0 - \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & I_n \end{bmatrix}\right)\begin{bmatrix} O & O \\ O & I_m \\ I_n & O \end{bmatrix} \\ &= -[G_{12}K_0(I - G_{22}K_0)^{-1} \quad G_{12}(I - K_0G_{22})^{-1}], \\ T_3 &:= \begin{bmatrix} O & O & I_n \\ O & I_m & O \end{bmatrix}\left(\Theta_0 - \begin{bmatrix} O & O & O \\ O & I_m & O \\ O & O & O \end{bmatrix}\right)\begin{bmatrix} I_{m'} \\ O \\ O \end{bmatrix} \\ &= \begin{bmatrix} (I - G_{22}K_0)^{-1}G_{21} \\ K_0(I - G_{22}K_0)^{-1}G_{21} \end{bmatrix}. \end{aligned}$$

Since every element of T_1, T_2, and T_3 is an element of Θ_0, the matrices T_1, T_2, and T_3 are over $\mathcal{A}$. We now see that the standard control problem is to find Q that minimizes the norm of $T_1 - T_2QT_3$, which is a model-matching problem. Once we know this Q by solving the model-matching problem, we can know the solution of the standard control problem, that is, the transfer matrix K by letting $K := \Omega_{21}\Omega_{22}^{-1}$, where Ω_{21} and Ω_{22} are the (2,1)- and (2,2)-blocks of $\Omega(Q)$ in (7). Note that because $-G_{22}$ as well as G_{22} is strictly causal, Ω_{22} is nonsingular and K is causal by Proposition 5 of [M02a]. □

5 Application

The result of this paper can be applied to models that can be expressed under the coordinate-free approach.

Let us consider the multidimensional systems with structural stability. To apply the coordinate-free approach, $\mathcal{A}$ and $\mathcal{Z}$ are given as follows:

$$\begin{bmatrix} O & G_{12} \\ O & I_m \\ I_n & O \end{bmatrix}\left(\left(H_0 - \begin{bmatrix} I_n & O \\ O & O \end{bmatrix}\right)Q\left(H_0 - \begin{bmatrix} O & O \\ O & I_m \end{bmatrix}\right) + H_0\right)\begin{bmatrix} G_{21} & O & I_n \\ O & I_m & O \end{bmatrix} + \begin{bmatrix} G_{11} & O & O \\ O & O & O \\ O & O & O \end{bmatrix} \quad (8)$$

$$\left(\Theta_0 - \begin{bmatrix} O & O & O \\ O & O & O \\ O & O & I_n \end{bmatrix}\right)\begin{bmatrix} O & O \\ O & I_m \\ I_n & O \end{bmatrix} Q \begin{bmatrix} O & O & I_n \\ O & I_m & O \end{bmatrix}\left(\Theta_0 - \begin{bmatrix} O & O & O \\ O & I_m & O \\ O & O & O \end{bmatrix}\right) + \Theta_0. \quad (9)$$

$$\mathcal{A} = \{a/b \mid a, b \in \mathbb{R}[z_1, \ldots, z_l],\ b \neq 0 \text{ in } \overline{U}^l\},$$
$$\mathcal{Z} = \sum_{i=1}^{l} z_i \mathcal{A} = \{a/b \in \mathcal{A} \mid a, b \in \mathbb{R}[z_1, \ldots, z_l], \text{ the constant term of } a \text{ is zero.}\},$$

where $\overline{U}^l$ denotes the closed unit polydisc.

So far it is not known yet whether or not all stabilizable plants of the multidimensional system have both right- and left-coprime factorizations[Lin01]. Even so, by the result of this paper, we can consider that the standard control problem and the model-matching problem are equivalent. This is an answer of the statement given by Ball and Malakorn in [BM02] (appeared in Section 1).

Next let us consider the neutral systems for fractional exponential delay systems [BP01]. This has analogous situation. The method to find doubly coprime factorizations for this model is still under study [BP01]. Even so, the coordinate-free approach can be applied to this model by letting $\mathcal{A}$ be $\widehat{\mathcal{A}}$ of [BP01] and $\mathcal{Z}$ consist of all transfer functions f in $\widehat{\mathcal{A}}$ such that $\lim_{s\to\infty} f$ is finite.

6 Conclusion

This paper has presented the equivalence between the standard control problem and the model-matching problem without the help of coprime factorization. It should be noted that the sizes of the parameter matrices of the Youla-Kučera parameterization [YJB76, K75] and the method of Section 3 are different; the later is larger than or equal to the former. The method to find the minimum number of the parameters should be investigated even though it is independent of the theme of this paper.

References

[A85] V. Anantharam. On stabilization and the existence of coprime factorizations. *IEEE Trans. Automat. Contr.*, AC-30:1030–1031, Oct. 1985.

[BM02] J.A. Ball and T. Malakorn. Feeback control for multidimensional systems and interpolation problems for multivariable functions. In D.S. Gillam and J. Rosenthal, editors, *Proc. 15th MTNS*, South Bend, IN, Aug. 2002.

[BP01] C. Bonnet and J.R. Partington. Stabilization of fractional exponential systems including delays. *Kybernetika*, 37(3):345–353, 2001.

[DFT92] J.C. Doyle, B.A. Francis, and A.R. Tannenbaum. *Feedback Control Theory*. New York, NY: Macmillan Publishing Company, 1992.

[DLMS80] C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks. Feedback system design: The fractional representation approach to analysis and synthesis. *IEEE Trans. Automat. Contr.*, AC-25:399–412, Mar. 1980.

[F87] B.A. Francis. *A Course in H_∞ Control Theory*, volume 88 of *Lecture Notes in Control and Information Sciences*. Berlin, Heidelberg: Springer-Verlag, 1987.

[K75] V. Kučera. Stability of discrete linear feedback systems. In *Proc. of the IFAC World Congress*, 1975. Paper No.44-1.

[Kwa85] H. Kwakernaak. Minimax frequency domain performance and robustness optimization of linear feedback systems. *IEEE Trans. Automat. Contr.*, AC-30:994–1004, Oct. 1985.

[Lin99] Z. Lin. Feedback stabilization of MIMO 3-D linear systems. *IEEE Trans. Automat. Contr.*, 44:1950–1955, Oct. 1999.

[Lin01] Z. Lin. Output feedback stabilizability and stabilization of linear n-D systems. In *Multidimensional Signals, Circuits and Systems*. New York, NY: Taylor & Francis, 2001.

[MA01] K. Mori and K. Abe. Feedback stabilization over commutative rings: Further study of coordinate-free approach. *SIAM J. Control and Optim.*, 39(6):1952–1973, 2001.

[M02a] K. Mori. Parameterization of stabilizing controllers over commutative rings with application to multidimensional systems. *IEEE Trans. Circuits and Syst. I*, 49:743–752, Jun. 2002.

[M02b] K. Mori. Parameterization of stabilizing controllers with either right- or left-coprime factorization. *IEEE Trans. Automat. Contr.*, pages 1763–1767, Oct. 2002.

[M03] K. Mori. Controller parameterization of anantharam's example. *IEEE Trans. Automat. Contr.*, pages 1655–1656, Sep. 2003.

[SS92] S. Shankar and V. R. Sule. Algebraic geometric aspects of feedback stabilization. *SIAM J. Control and Optim.*, 30(1):11–30, 1992.

[S94] V. R. Sule. Feedback stabilization over commutative rings: The matrix case. *SIAM J. Control and Optim.*, 32(6):1675–1695, 1994.

[VJ84] M. Verma and E. Jonckheere. L_∞-compensation with mixed sensitivity as a broadband matching problem. *System and Control Letters*, 4:125–130, May 1984.

[VSF82] M. Vidyasagar, H. Schneider, and B. A. Francis. Algebraic and topological aspects of feedback stabilization. *IEEE Trans. Automat. Contr.*, AC-27:880–894, Apr. 1982.

[YJB76] D.C. Youla, H.A. Jabr, and J.J. Bongiorno, Jr. Modern Wiener-Hopf design of optimal controllers, Part II: The multivariable case. *IEEE Trans. Automat. Contr.*, AC-21:319–338, Mar. 1976.

[Zam81] G. Zames. Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses. *IEEE Trans. Automat. Contr.*, AC-26:301–320, Apr. 1981.

[ZDG96] K. Zhou, J.C. Doyle, and K. Glover. *Robust and Optimal Control.* Upper Saddle River, NJ: Prentice-Hall, 1996.

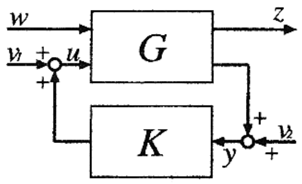

Figure 1: Standard Control Problem

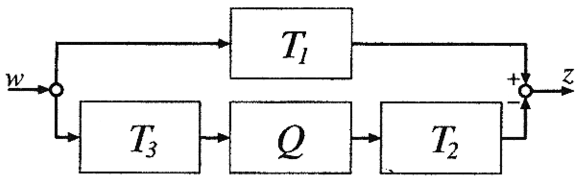

Figure 2: Model-Matching Problem

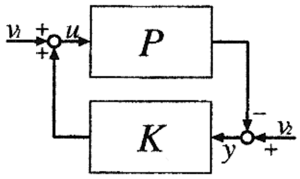

Figure 3: Feedback System