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厳密にプロパーな補償器のパラメトリゼーション
Parametrization of All Strictly Proper Controllers

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Abstract

We give a parameterization of all stabilizing controllers with some fixed precompensator for single-input single-output systems. The framework we use is the factorization approach, By using this parametrization, we give a parameterization of all strictly causal stabilizing controllers.

1 Introduction

It is well known that the factorization approach to control systems has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, one-dimensional as well as multidimensional systems, etc.[1, 5]. Hence the result given in this paper will be able to a number of models. In factorization approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. This approach leads to conceptually simple and computationally tractable solutions to many important and interesting problems[4]. A transfer function of this approach is considered as the ratio of *two* stable causal transfer functions. Further the set of the stable causal transfer functions is considered as a *commutative ring*.

The choice of the stabilizing controller is important for the resulting closed loop because, in general, the stabilizing controllers are not unique. In the classical case, the stabilizing controllers can be parameterized by the method called Youla-Kučera-parameterization[1, 2, 3, 4, 5, 6]. However, it is also known that such parametrization may include a stabilizing controllers which is not causal and may result a direct loop (see Section 2).

The objective of this paper is to present an alternative parametrization of stabilizing controllers, that is the parametrization of all *strictly causal* stabilizing controllers. This parametrization will include neither any non-causal stabilizing controller nor any direct loop.

2 Motivation

Let us consider the feedback system shown in Figure 1 of the classical discrete-time system. Let

$$\mathcal{A} = \{f(z) \in \mathbb{R}(z) \mid f(z) \text{ has no poles on or inside the unit circle in the complex plane}\} \quad (1)$$

(z denotes the delay operator), which is the set of stable causal transfer functions.

Now we let $p = z + 1$. Then we have the Bézout identity $(z+1)y + x = 1$ over $\mathcal{A}$. Using the Youla-Kučera-parameterization, these y and x can be parametrized as $y = r$, $x = 1 - (z+1)r$ with a parameter r of $\mathcal{A}$.

By letting $r = 1$, we have $y = 1$ and $x = -z$. In this case, the stabilizing controller obtained from y and x is $-1/z$. Unfortunately this is not causal.

Otherwise, by letting $r = -1$, we have $y = -1$ and $x = z + 2$. In this case, the stabilizing controller obtained from y and x is $-1/(z+2)$. This results a direct loop, that is, the current input affects the whole closed feedback system immediately. This loop is normally unsafe even if the total feedback system is stable.

To avoid these situation, we will present the parametrization of all strictly causal stabilizing controllers, (i) which does not include any non-causal stabilizing controller and (ii) which does not result any direct loop.

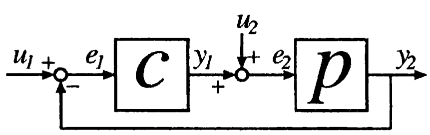

Figure 1: Feedback system Σ.

3 Preliminaries

We employ the factorization approach [1, 3, 4, 5] and the symbols used in [7] and [8].

Denote by $\mathcal{A}$ a unique factorization domain that is the set of stable causal transfer functions. The total field of fractions of $\mathcal{A}$ is denoted by $\mathcal{F}$; that is,

$$\mathcal{F} = \{n/d \mid n, d \in \mathcal{A},\ d \neq 0\}.$$

This $\mathcal{F}$ is considered to be the set of all possible transfer functions. Let $\mathcal{Z}$ be a prime ideal of $\mathcal{A}$ with $\mathcal{Z} \neq \mathcal{A}$. Further, let

$$\begin{aligned} \mathcal{P} &= \{a/b \in \mathcal{F} \mid a \in \mathcal{A},\ b \in \mathcal{A} - \mathcal{Z}\} \text{ and} \\ \mathcal{P}_s &= \{a/b \in \mathcal{F} \mid a \in \mathcal{Z},\ b \in \mathcal{A} - \mathcal{Z}\}. \end{aligned}$$

A transfer function f is said to be *causal* (*strictly causal*) if and only if f is in $\mathcal{P}$ ($\mathcal{P}_s$).

We consider the feedback system Σ [4, Ch.5, Figure 5.1] shown in Figure 1. In the figure, p denotes a *plant* in $\mathcal{P}$ and c a *controller*. The stabilization problem, considered in this paper, follows the one developed in [1, 4, 5]. For details, the reader is referred to [4, 9, 8, 10].

Let $H(p,c)$ denote the transfer matrix from $[\,u_1^t \quad u_2^t\,]^t$ to $[\,e_1^t \quad e_2^t\,]^t$ of the feedback system Σ, that is,

$$H(p,c) := \begin{bmatrix} (1+pc)^{-1} & -p(1+pc)^{-1} \\ c(1+pc)^{-1} & (1+pc)^{-1} \end{bmatrix} \qquad (2)$$

($\in \mathcal{F}^{2\times 2}$) provided that $1+pc$ is nonzero. We say that the plant p is *stabilizable*, p is *stabilized* by c, and c is a *stabilizing controller* of p if and only if $1+pc$ is nonzero and $H(p,c) \in \mathcal{A}^{2\times 2}$. In the definition above, we do not mention the causality of the stabilizing controller. However, it is known that if a causal plant is stabilizable, there always exists a causal stabilizing controller of the plant [9].

A pair a and b of $\mathcal{A}$ are said to be *coprime* (over $\mathcal{A}$) if and only if there exist x and y of $\mathcal{A}$ such that $xa + yb = 1$ holds. An ordered pair n and d of $\mathcal{A}$ are said to be a *coprime factorization* of p if and only if (i) d is nonzero, (ii) $p = n/d$ over $\mathcal{F}$, and (iii) n and d are coprime[1, 4, 5].

A pair a and b of $\mathcal{A}$ are said to be *factor coprime* (over $\mathcal{A}$) if and only if the following holds: for any x of $\mathcal{A}$, if x divides both a and b, then x is a unit of $\mathcal{A}$. A pair a and b of $\mathcal{F}$ are said to be *rationally factor coprime* (over $\mathcal{F}$) if and only if there exist x_1, y_1, x_2, y_2 of $\mathcal{A}$ such that (i) $a = y_1/x_1$ and $b = y_2/x_2$, (ii) y_1 and x_2 are factor coprime, and (iii) y_2 and x_1 are factor coprime.

Because we investigate the set of some kind of stabilizing controllers, we introduce some notations as follows ($p \in \mathcal{P}$):

$$\begin{aligned} \mathcal{S}(p) &:= \{c \in \mathcal{F} \mid H(p,c) \in \mathcal{A}^{2\times 2}\}, \\ \mathcal{SP}(p) &:= \mathcal{S}(p) \cap \mathcal{P}, \\ \mathcal{SP}_s(p) &:= \mathcal{S}(p) \cap \mathcal{P}_s. \end{aligned}$$

In this paper, we consider a fixed causal *precompensator* ζ ($\in \mathcal{P}$) as a part of a controller c as shown in Figure 2 ($c = c_0\zeta$). We assume that c_0 and ζ must be rationally factor coprime.

We further introduce the set of all (causal) stabilizing controllers of p including a precompensator as follows ($p, \zeta \in \mathcal{P}$):

$$\begin{aligned} \mathcal{S}(p;\zeta) &:= \{c_0\zeta \mid c_0 \in \mathcal{F},\ c_0\zeta \in \mathcal{S}(p), \\ &\qquad c_0 \text{ and } \zeta \text{ are rationally} \\ &\qquad \text{factor coprime}\} \qquad (3) \\ \mathcal{SP}(p;\zeta) &:= \{c_0\zeta \mid c_0 \in \mathcal{P},\ c_0\zeta \in \mathcal{SP}(p), \\ &\qquad c_0 \text{ and } \zeta \text{ are rationally} \\ &\qquad \text{factor coprime}\} \qquad (4) \end{aligned}$$

For the notion of rational factor coprimeness, we have the following proposition.

Proposition 1 *Let a and b be elements of $\mathcal{F}$. Suppose that a and b are rationally factor coprime. Then ab is in $\mathcal{A}$ if and only if both a and b are in $\mathcal{A}$.*

Proof. "If" part is obvious. Hence we show "Only if" part only.

Let a_n, a_d, b_n, b_d be in $\mathcal{A}$ with $a = a_n/a_d$ and $b = b_n/b_d$ such that each of pairs (a_n, a_d), (b_n, b_d), (a_n, b_d) and (b_n, a_d) is factor coprime. Suppose now that ab is in $\mathcal{A}$. Then a_d is a unit of $\mathcal{A}$ because the pairs (a_n, a_d) and (b_n, a_d) are factor coprime. This means that a is in $\mathcal{A}$. Analogously b is also in $\mathcal{A}$. □

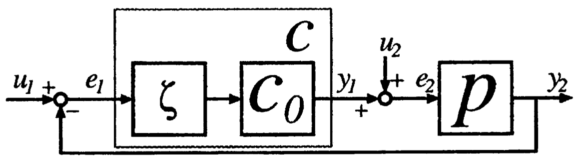

Figure 2: Feedback system with a precompensator.

4 All strictly causal stabilizing controllers

In this section, we give the parameterization of all strictly causal stabilizing controllers provided that $\mathcal{Z}$ is a principal ideal. In this case, the precompensator ζ is a generator of $\mathcal{Z}$. The following are primary results of this paper.

Theorem 1 *Let p and ζ be elements of $\mathcal{P}$. Assume that p and ζ are rationally factor coprime. If $H(p\zeta, c)$ is over $\mathcal{A}$, then $H(p, c\zeta)$ is over $\mathcal{A}$.*

Theorem 2 *Let p and ζ be elements of $\mathcal{P}$. Suppose that p and ζ are rationally factor coprime. Then the following (i) and (ii) hold.*

(i)

$$\mathcal{S}(p;\zeta) = \{c_0\zeta \mid c_0 \in \mathcal{S}(p\zeta)\}. \quad (5)$$

(ii) *Let n, d, y, x be elements in $\mathcal{A}$ with $p\zeta = n/d$, $c = y/x$ such that $ny + dx = u$, where u is a unit of $\mathcal{A}$. Then*

$$\mathcal{S}(p;\zeta) = \{\zeta\frac{y+rd}{x-rn} \mid r \in \mathcal{A},\ x - rn \neq 0\}. \quad (6)$$

Theorem 3 *Let p be a stabilizable plant of $\mathcal{P}$. Assume that $\mathcal{Z}$ is in a principal ideal and its generator is ζ.*

(i) If $p\zeta$ is not stabilizable, then $\mathcal{SP}_s(p) = \emptyset$.

(ii) Otherwise, there exist n, d, y and x be of $\mathcal{A}$ such that $p\zeta = n/d$ and $ny + dx = u$, where u is a unit of $\mathcal{A}$. Then the set of all strictly causal stabilizing controllers of p is given as follows:

$$\mathcal{SP}_s(p) = \{\zeta\frac{y+rd}{x-rn} \mid r \in \mathcal{A}\}. \quad (7)$$

In the following, we give the proof of Theorem 1 only. The other two proofs for Theorems 2 and 3 are omitted.

Proof. We first consider two cases: (i) $p\zeta = 0$ and (ii) $p\zeta \neq 0$.

(i) $p\zeta = 0$. Since p and ζ are rationally factor coprime, p and ζ are in $\mathcal{A}$ by Proposition 1. We here consider further two cases: (i-1) $p = 0$ and (i-2) $p \neq 0$.

(i-1) $p = 0$. Because $H(0,c)$ is over $\mathcal{A}$, c is in $\mathcal{A}$. Since both ζ and c are in $\mathcal{A}$, $H(0,c\zeta)$ is still over $\mathcal{A}$.

(i-2) $p \neq 0$. Then ζ is equal to 0. Because of $p \in \mathcal{A}$, $H(p,0)$ $(= H(p,c\zeta))$ is over $\mathcal{A}$.

(ii) $p\zeta \neq 0$ (that is, $p \neq 0$ and $\zeta \neq 0$). We consider further two cases: (ii-1) $c = 0$ and (ii-2) $c \neq 0$.

(ii-1) $c = 0$. Suppose that $H(p\zeta,0)$ is over $\mathcal{A}$. Then $p\zeta$ itself is in $\mathcal{A}$ from the $(1,2)$-entry of $H(p\zeta,0)$. By Proposition 1, p is in $\mathcal{A}$. Hence $H(p,0)$ is over $\mathcal{A}$.

(ii-2) $c \neq 0$. Now, all of p, c, and ζ are nonzero. Suppose that $H(p\zeta,c)$ is over $\mathcal{A}$. Let n, d, y and x be elements in $\mathcal{A}$ with $p\zeta = n_0/d_0$ and $c = y/x$ such that

$$n_0y + d_0x = 1 \quad (8)$$

(This Bézout identity exists from Corollary 2.1.5 of [10]). Because p and ζ are rationally factor coprime, there exist n, d, ζ_n, and ζ_d in $\mathcal{A}$ such that $p = n/d$, $\zeta = \zeta_n/\zeta_d$, $n_0 = n\zeta_n$, $d_0 = d\zeta_d$, and (n,ζ_d) and (d,ζ_n) are factor coprime. Then from (8) we have

$$ny\zeta_n + dx\zeta_d = 1$$

and

$$\begin{aligned} H(p,c\zeta) &= \begin{bmatrix} (1+pc\zeta)^{-1} & -p(1+pc\zeta)^{-1} \\ c\zeta(1+pc\zeta)^{-1} & (1+pc\zeta)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} dx\zeta_d & nx\zeta_d \\ dy\zeta_n & dx\zeta_d \end{bmatrix}, \end{aligned}$$

which is over $\mathcal{A}$. □

Based on Theorem 3, we obtain the parametrization of all strictly causal stabilizing controllers. In the following, we consider the parametrization of all strictly causal stabilizing controllers of two classical models (the continuous-time system and the discrete-time system).

4.1 Classical Continuous-Time System

Consider the classical continuous-time systems. Let C_+ denote the closed right half-plane $\{s \mid \Re s \geq 0\}$ and C_{+e} denote the extended right half-plane, that is, C_+ together with the point at infinity. Then the set $\mathcal{A}$ of stable causal transfer functions is given by

$$\mathcal{A} = \{f(s) \in \mathbb{R}(s) \mid \sup_{s\in C_{+e}} |f(s)| < \infty\}.$$

It is known that this $\mathcal{A}$ is a Euclidean domain with the degree function $\delta : (\mathcal{A} - \{0\}) \to \mathbb{Z}_+$:

$$\delta(f) = \text{“number of zeros of } f \text{ in } C_{+e}\text{”}$$

(See Chapter 2 of [4]). The ideal $\mathcal{Z}$ for the definition of the causality is given as

$$\mathcal{Z} = \{f \in \mathcal{A} \mid\ f = n/d,\ n,d \in \mathbb{R}[s],\ \deg(n) < \deg(d)\},$$

which is a prime and principal ideal. In fact, for f in $\mathcal{A}$, the ideal (f) is equal to $\mathcal{Z}$ if and only if $\delta(f) = 1$ and $\deg(n) < \deg(d)$, where n and d are polynomials of s over $\mathbb{R}$ with $f = n/d$. The generator of $\mathcal{Z}$ can be, for example,

$$\frac{1}{s+1}, \frac{-1}{s+2}, \frac{s+3}{(s+1)(s+2)}, \frac{s+5}{s^2+2s+2},$$

and so on.

Example 1 Let

$$p = s/(s-1).$$

Consider to obtain the set $\mathcal{SP}_s(p)$ of all strictly causal (proper) stabilizing controllers. First consider

$$\mathcal{Z} = (\frac{1}{s+1}).$$

Thus, let

$$\zeta = 1/(s+1).$$

Then

$$p\zeta = s/((s-1)(s+1)).$$

We have

$$ny + dx = u,$$

where

$$\begin{aligned} &p\zeta = n/d, \\ &n = \frac{s}{(s+1)^2}, \quad d = \frac{s-1}{s+1}, \\ &y = \frac{2(s+2)}{s+1}, \quad x = \frac{s-0.5}{s+1}, \\ &u = \frac{s^3 + 1.5s^2 + 3s + 0.5}{(s+1)^3}. \end{aligned}$$

Then u is a unit of $\mathcal{A}$ because the zeros of u are

$$-0.659 \pm 1.525i \text{ and } -0.1810.$$

Now $\mathcal{SP}_s(p)$ is given as in (7) by virtue of Corollary 3. For example, letting

$$r = 7/(s+2),$$

we obtain the following stabilizing controller:

$$\frac{4s^2 + 30s + 2}{2s^3 + 5s^2 - 13s - 2},$$

which is strictly causal. ■

4.2 Classical Discrete-Time System

Consider next the classical discrete-time systems. In this case, the set $\mathcal{A}$ of stable causal transfer functions is given as (1). It is also known that this $\mathcal{A}$ is a Euclidean domain with the degree function $\delta : (\mathcal{A} - \{0\}) \to \mathbb{Z}_+$:

$\delta(f) =$ "number of zeros of f inside the close unit circle"

(See again Chapter 2 of [4]). The ideal $\mathcal{Z}$ for the definition of the causality is given as

$$\mathcal{Z} = \{f \in \mathcal{A} \,|\, f = zf_0, \ f_0 \in \mathcal{A}\},$$

which is obviously a prime and principal ideal. In fact, for f in $\mathcal{A}$, the ideal (f) is equal to $\mathcal{Z}$ if and only if $f = zf_0$, where f_0 is a unit of $\mathcal{A}$. The generator of $\mathcal{Z}$ can be, for example,

$$z, \ (z^2 + 2z)/(z+3),$$

and so on.

Example 2 Let us consider the plant in Section 2, that is, $p = z + 1$. Analogously to Example 1, consider again to obtain the set $\mathcal{SP}_s(p)$ of all strictly causal stabilizing controllers. Let $\zeta = z$, that is,

$$\mathcal{Z} = z\mathcal{A}.$$

Then $p\zeta = z(z+1)$. We have

$$ny + dx = 1,$$

where

$$\begin{aligned} &p\zeta = n/d, \\ &n = z(z+1), \ d = 1, \\ &y = 0, \ x = 1. \end{aligned}$$

Now $\mathcal{SP}_s(p)$ is given as in (7). For example, letting $r = 7/(z+2)$, we obtain the following stabilizing controller:

$$\frac{-7z}{7z^2 + 6z - 2},$$

which is not stable but is strictly causal. ■

5 Conclusion and Future Works

In this paper, we have given the parameterization of all strictly causal stabilizing controllers. Other applications of the method of this paper will include (i) the parameterization of all causal stabilizing controllers including the integrator and (ii) for multidimensional systems, the parameterization of all stabilizing controllers including all delay operators, which will be presented in the future.

References

[1] C. A. Desoer, R. W. Liu, J. Murray, and R. Saeks, "Feedback system design: The fractional representation approach to analysis and synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 399–412, 1980.

[2] V. Kučera, "Stability of discrete linear feedback systems," in *Proc. of the IFAC World Congress*, 1975, Paper No.44-1.

[3] V.R. Raman and R. Liu, "A necessary and sufficient condition for feedback stabilization in a factor ring," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 941–943, 1984.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*, Cambridge, MA: MIT Press, 1985.

[5] M. Vidyasagar, H. Schneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.*, vol. AC-27, pp. 880–894, 1982.

[6] D.C. Youla, H.A. Jabr, and J.J. Bongiorno, Jr., "Modern Wiener-Hopf design of optimal controllers, Part II: The multivariable case," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 319–338, 1976.

[7] K. Mori, "Parameterization of stabilizing controllers over commutative rings with application to multidimensional systems," *IEEE Trans. Circuits and Syst. I*, vol. 49, pp. 743–752, 2002.

[8] V. R. Sule, "Feedback stabilization over commutative rings: The matrix case," *SIAM J. Control and Optim.*, vol. 32, no. 6, pp. 1675–1695, 1994.

[9] K. Mori and K. Abe, "Feedback stabilization over commutative rings: Further study of coordinate-free approach," *SIAM J. Control and Optim.*, vol. 39, no. 6, pp. 1952–1973, 2001.

[10] S. Shankar and V. R. Sule, "Algebraic geometric aspects of feedback stabilization," *SIAM J. Control and Optim.*, vol. 30, no. 1, pp. 11–30, 1992.

[11] K. Mori, "Elementary proof of controller parametrization without coprime factorizability," *IEEE Trans. Automat. Contr.*, vol. AC-49, pp. 589–592, 2004.