# Scientific Study of Road Traffic Flow（1） <br> －Road Traffic Fundamentals from Weber－Fechner Law－ <br> －Tasuku Takagi <br> Professor Emeritus，Tohoku University 

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## 1．Introduction

In 1930s，the road traffic phenomena came into a new research genre，and many papers and books were published．Today，as the motorizations have been globally expanding，the problems concerning road traffic have been more and more serious，which are symbolized as the environmental hazards and economical losses caused by heavy traffic congestions．The most terrible is that the developments of motorization in the world have been progressed without basic or fundamental countermeasures to cope with those problems．The big project of ITS（Intelligent Transportation System）is now on the way，of which target is concentrated ultimately to the congestion problems．Although the problems have gradually been improved by ITS technologies concerning high technological information and communication systems，we cannot fully satisfy today＇s situation．The reason is that we have not a reliable fundamental
theory to cope with the problems．We should find the basic countermeasures for coping with those problems with more scientific approaches．
In the tremendous amount of past research works，few have been seen from the viewpoint of pure science．Particularly，we should notice that the moving cars on road are maneuvered by the drivers．However，past studies on the traffic flow have ignored this important point of human（driver）factors，and almost all researchers have treated the traffic flow as a physical phenomenon ignoring the driver＇s sensation．

If we take the human（driver＇s）factors into road traffic flow analyses，different approaches and practices can be expected．
The Weber－Fechner Law shows the mathematical relationship between physical stimuli to human sensory system and human sensation．In this article，as a new finding，we
will see the Weber-Fechner Law is applicable to the driver's sensation. And basing upon this new evidence, the traffic flow can be discussed from both points of statistics and time domain behaviors.

The statistical flow analyses are commonly carried out so far. Historically, $\mathrm{k}-\mathrm{V}$ and $\mathrm{k}-\mathrm{q}$ relationship were commonly discussed, where k in this case is a vehicle density (number of vehicles per 1 km ), V is speed ( $\mathrm{km} / \mathrm{h}$ ), and q is a flow (number of vehicles per 5 min . 10 min . 1hour, etc.). Those statistical analyses mentioned above have not practically been used for prevention of traffic congestion. The reason is naturally understandable, because the traffic flow phenomena are the time domain ones.

Since the statistical analysis characterizes the accumulated data with respect to time, it has no ability to extract the time sequential characteristics.

The above mentioned will be concretely mentioned in this study along with the actual measured data.

In this study, a road traffic flow is analyzed by measuring T and V , where $\mathrm{T}(\mathrm{s})$ denotes a time-headway (time interval) of each paired cars that subsequently pass the designated point on road, and $\mathrm{V}(\mathrm{m} / \mathrm{s})$ each car speed.

We will see any road traffic flows obey the Weber-Fechner law between X (= VT) and V. In this case, X which is recognized by driver's eyes, can be considered as physical stimulus, and V as driver's sensation. The discovery of Weber-Fechner law in road traffic flow made the essential apprehensions about the traffic flow such as how to specify the road quality of easiness for running.
Time domain analysis is not common so far in the traffic flow abalyses, but since all traffic flows exist in the time domain, the all problems
concerning traffic flow should be treated on the time domain. One of the most annoying problems is traffic congestion which is a typical example of time domain phenomena. We want to know the details of traffic congestion in conjunction with flow dynamics. If we could forecast the congestion before some amount of time, there should be unfathomable effect on our social activities. If we could have more sophisticate way that controls the urban traffic system together with the ITS technologies, its effects on our economical societies must be great.

The scientific approaches which will be mentioned in this study are unique in the meaning that has never been seen in the past. Any ideas without scientific ground were avoided in carrying out for developing the theory. Every idea has been supported by the actual measured data. All data (T and V) were taken from actual video images which were taken at different road including expressway, urban and suburban road. As a matter of course, urban and suburban roads have traffic signals. The time domain investigations include the effects of traffic control signals. Before entering the theory, we shall see the measured T and V , then $\mathrm{X}(=\mathrm{VT})$ with respect to time t .

## 2. Weber-Fechner Law

Relationship between physical stimuli and sensation can be written as a mathematical formula that is called Weber-Fechner Law. Ernst Heinrich Weber (1795-1878) investigated about human sensation against physical stimuli which is applied to human sensory system. His experimental results have been written like

$$
\begin{equation*}
\mathrm{k} \frac{\Delta \mathrm{~W}}{\mathrm{~W}}=\Delta \mathrm{S} \tag{2.1}
\end{equation*}
$$

where k is a constant, W and $\Delta \mathrm{W}$ denote
physical stimuli and its increment of change, respectively, and $\Delta \mathrm{S}$ was denoted as a sensation threshold. This expression (2.1) is called Weber's formula.

We are able to understand (2.1) as follows:
In case of weight of bucket water which a blindfolded person to be tested is holding, the initial stimulus (weight) is W. Water is gradually poured into the bucket. The person under test is asked in advance to make a sign when he noticed the increase of weight. The increment weight of water is $\Delta \mathrm{W}$. It is clear that one cannot sense it when the weight of poured water is so small.

The conclusions obtained from experiments by Weber are summarized as follows:

1) When the value of $\Delta W / W$ reached some constant value, sign was given for each experiment. This meant our sensation can respond to relative increase of stimulus ( $\Delta \mathrm{W} / \mathrm{W}$ ), not to direct increase $\Delta \mathrm{W}$.
2) Sensation has some threshold to sense the stimulus variation. This threshold was denoted as $\Delta \mathrm{S}$.
3) Different coefficient k was necessary for each experiment on a different type of stimuli.
The above mentioned threshold in 2) is that of our feeling sensation. Physiologically, the nerve can sense very small changes of weight if the stimulus change was above the nerve impulse firing threshold which is very small compared with $\Delta \mathrm{S}$. From this idea, Weber's Law (2.1) can be written like

$$
\mathrm{k} \frac{\mathrm{dW}}{\mathrm{w}}=\mathrm{dS}
$$

where $\Delta \mathrm{S}$ in (2.1) becomes dS, nerve firing threshold, which is much smaller than $\Delta S$ and can be seen infinitesimally so small that we can see $S$ is continuous.

Gustav Theodor Fechner (1801-1887) showed the sensation is logarithmically proportional to the stimulou level, and derived the formula

$$
\begin{equation*}
\mathrm{k} \ln \left(\frac{\mathrm{w}}{\mathrm{~W}_{0}}\right)=\mathrm{S} \tag{2.3}
\end{equation*}
$$

where $\ln$ denotes natural $\log$ and $\mathrm{W}_{0}$ is W when $S=0$. Clearly, (2.3) is the result of integral form of (2.2). (2.3) can be written as an exponential form like

$$
\begin{equation*}
\mathrm{W}=\mathrm{W}_{0} \exp \left(\frac{1}{\mathrm{k}} \mathrm{~S}\right) \tag{2.4}
\end{equation*}
$$

Since the expression (2.4) is comprehensive for discussing the present subject on the road traffic flow, we use this formula in this article, and we call both (2.3) and (2.4) Fechner's law

## 3. Weber-Fechner Law in Road Traffic Flow

### 3.1 Human factors in road traffic flow

Road traffic flow phenomena are not only physical ones, but belong to behavioral phenomena of human being who drives a car. A tremendous amount of paper and book has ever been published concerning the items on road traffic behaviors. But they have not been well applied in practical to the actual control of traffic flows such as signal lights at intersections, to which we feel that there exist much discrepancies between practical road managements and research works. The reason why, and from where, the above discrepancies came is the item we should begin to sincerely consider. It is clear that, since the past research works on the traffic flow have been carried out only from a viewpoint of physical phenomena, we are able to understand the reasons of the discrepancies mentioned above as follows: The traffic flow cannot be limited only to the physical phenomena, but it inevitably includes the human factors. When he or she drives a car, they always become nervous and cannot avoid tension being imposed on them for
maintaining the safety drive. The tension comes from the stimulus which almost all originates from driver's eyes that looks everything around. The most important stimulus is the eye-measured-distance between foregoing car and his or her own car. The driver does not vaguely look the forward road and foregoing car while they are driving. When the distance between cars (foregoing and own) becomes close, the driver instinctively control the own car speed so as to maintain the safety distance. This drive's action and relating behaviors are not pure physical but psychophysical phenomena. Thus, when we deal with the road traffic flow, we have to take into account the human factors.
Here we are assuming that the eye measured quantity of distance-headway (denoted by X ) is the stimulus imposed to the driver, and speed V the sensation. This assumption may be controversial because no ways exist to proof the idea. But so far as the speed $V$ is concerned, we consent that the driver cannot sense how fast the car is running if he or she does not look at the speed meter. This means that the speed as a physical quantity in the unit of $\mathrm{km} / \mathrm{h}$ or $\mathrm{m} / \mathrm{s}$ is less needed for driver to recognize. But since the speed is one of the most important parameters for the driver, a speed indicator (speed meter) is installed in front of the driver. On the contrary, eye measured distance quantity causes driver's tension and driver control the speed appropriately not to bump into the rear of foregoing car and keep safety to avoid any accidents.

### 3.2 Car following model

The fundamental idea in making model for analysis of a road traffic flow should be based upon the parameters X and V , because they each other have the relationship of the

Weber-Fechner Law as we will see later on.
In order to make the item comprehensive and to pursue the theoretical discussions, we shall begin from the car following model analysis. Fig. $3-1$ is the most essential model depicted through this study where the model in Fig.3-1 satisfies the conditions of necessity and sufficiency for discussing the subjects of our targeted items.
We shall see the detail of this model. The road traffic flow can be symbolized by two cars; Fig.3-1 shows the two cars (foregoing car A and its following one B ) at the different timings ( $\mathrm{t}_{1}$, $t_{2}, t_{3}$. The speed of each car denoted as $V_{A}$ and $V_{B}$, respectively. Fig.3-1 indicates that, at the timing of $t_{1}$, car A touches the designated point $D_{1}$. At the timing of $t_{2}, B$ touches the same point of $D_{1}$, and at the timing of $t_{3}$, A reaches the point $D_{2}$. Then $t_{2}-t_{1}=T_{A B}$ is the time difference between $A$ and $B$, and $T_{A B}$ is called time-headway between A and B. During TAB, the car A earns the run-distance of $\mathrm{X}_{\mathrm{AB}}$ that can be calculated by

$$
\begin{equation*}
X_{A B}=V_{A} T_{A B} \tag{3.1}
\end{equation*}
$$



Fig.3-1 Positions of vehicle A and B , and relationship between their speeds and distance-headway $\mathrm{X}\left(=\mathrm{X}_{\mathrm{AB}}\right)$ (1) : timing $\mathrm{t}_{1}$, (2) : timing $\mathrm{t}_{2}$, (3) : timing $\mathrm{t}_{3},\left(\mathrm{t}_{1}<\mathrm{t}_{2}<\mathrm{t}_{3}\right)$

The distance from head of foregoing car and that of following one is called distance-headway
between car A and B .
$X_{A B}$ shown in Fig. $3-1$ is the difference between run-distances made by $A$ within $X_{A B}$ and that by $B$, respectively. $\Delta X\left(=\Delta X_{A B}\right)$ becomes non-zero when the speed difference exists between $V_{A}$ and $V_{B}$, that is $V_{A}-V_{B}=V_{A B}$, and

$$
\begin{equation*}
\Delta \mathrm{X}_{\mathrm{AB}}=\mathrm{T}_{\mathrm{AB}} \Delta \mathrm{~V}_{\mathrm{AB}} \tag{3.2}
\end{equation*}
$$

### 3.3 Relationship between $X$ and $V$ as Weber-Fechner Law

In order to discuss the general traffic flow we need not specify vehicles like $A$ or $B$, the suffixes $A, B$ and $A B$ can be omitted and we can write (3.1) and (3.2) as follows, respectively;

$$
\begin{equation*}
\mathrm{X}=\mathrm{TV} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta \mathrm{X}=\mathrm{T} \Delta \mathrm{~V} \tag{3.2}
\end{equation*}
$$

As mentioned in 3.1, we assumed that $X$, the distance-headway, can be seen stimulus and $V$, the speed, sensation. Since we admit the relationship between X and V can obey the Weber-Fechner Law, we can write the mathematical formula similar to the formula (2.1) in Section 2, and we can write like

$$
\begin{equation*}
\frac{\Delta x}{x}=\beta \Delta V \tag{3.3}
\end{equation*}
$$

where $\beta$ is used here in convenience instead of k of (2.1) in Section 2, and k was substituted by $1 / \beta$.
The validity of (3.3) should be verified. One way of verification is mathematical approach, and the other one is that from actual traffic measurements which will be mentioned in Section 4 (to be presented as "Scientific Study of Road Traffic Flow (2)" which will be next presentation in (2) in this series).

### 3.4 Mathematical Verification of $\mathbf{V}-\mathbf{X}$ Weber-Fechner Relationship

If we looking at formulae (3.1) and (3.2), $\Delta \mathrm{X} / \mathrm{X}$
in (3.3) becomes

$$
\begin{equation*}
\frac{\Delta X}{X}=\frac{T \Delta V}{T V}=\frac{\Delta V}{V} \tag{3.4}
\end{equation*}
$$

then (3.3) becomes

$$
\begin{equation*}
\frac{\Delta \mathrm{V}}{\mathrm{~V}}=\beta \Delta \mathrm{V} \tag{3.5}
\end{equation*}
$$

This is primitive. The answer is

$$
\begin{equation*}
\mathrm{V} \beta=1 \tag{3.6}
\end{equation*}
$$

Under the condition of (3.6), equation (3.3) always comes into being. Thus we can mathematically admit that the formula (3.3) shows the Weber-Fechner Law in road traffic flow. However, this should be verified from the actual measured results as will be mentioned in Section 4.

- Note that the mathematical formula shown like (3.3) not always shows the Weber's law. This type formula is a common one in the basic equation (differential formula) from which the exponential equation is derived. More important is the fact that the exponential function is very commonly seen in natural as shown in Appendix 3-1 attached the end of this Section as an example of the edge of the Mt. Fuji (See Appendix 3-1).


### 3.5. Fechner's Law in Road Traffic Flow and Meaning of $\boldsymbol{\beta V}=\mathbf{1}$

We have shown in Section 2 the Fechner's law which was shown by the logarithmic formula or the exponential one. Hereafter, we use the exponential formula for convenience which is written as, developing from (2.4) in Section 2,

$$
\begin{equation*}
X=X_{0} \exp (\beta V) \tag{3.7}
\end{equation*}
$$

where $X_{0}$ is the value of $X$ at $V=0$ which is the distance-headway at the speed $V=0$. The situation of $V=0$ can be seen at any intersections as a queue of cars at red signal light. According to the observations, $X_{0}$ drops
from 7 to 15 m , and we take $\mathrm{X}_{0}=10 \mathrm{~m}$ as an average value for the time being.
We should get the answer of $\beta V=1$ of (3.6), what does this condition means? One solution can be obtained from the Fechner's law of formula (3.7), as follows: Since $\mathrm{X}=\mathrm{TV}$, the time-headway T can be derived

$$
\begin{equation*}
T=\frac{X_{0}}{V} \exp (\beta V) \tag{3.8}
\end{equation*}
$$

Since this function is concave with respect to V , the minimum value of $T$ (denoted by $\mathrm{T}_{\text {min }}$ ) is obtained at $V=1 / \beta \quad(\beta V=1$ from the solution of $\mathrm{dT} / \mathrm{dV}=0$ ) (see Appendix 3-2). That is to say, at the condition of $\beta V=1$, the maximum flow is obtained, and if we define the instantaneous flow rate q as $1 / T, T_{\text {min }}$ gives the maximum flow rate of $q_{\text {max }}$ at the speed $V_{\beta}(=1 / \beta)$, where $V_{\beta}$ is defined when $V$ is equal to $1 / \beta$.
We could see the meaning of $\beta V=1$ in the traffic flow as mentioned above, however, we cannot still understand the reason why the Weber's law in (3.3) should come into being under the particular condition of $\beta V=1$ of which answer is not simple, but we shall see from more sophisticate further discussions that will be done in the other Sections (will appear from Section 4 in this series).

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## Appendix 3-1

Analysis of the edge slope of Mount Fuji was carried out, and it could be expressed by the exponential function as follows:
Let R be the distance from arbitrary point of edge foot of the slope, and $L$ be the height at any position of $R$. Their increments are $\Delta R$ and $\Delta \mathrm{L}$ respectively as shown in the Fig. Ap3-1.
If we examine the relationship between $\Delta \mathrm{L} / \mathrm{L}$ and $\Delta R$ by using a ruler, we find the existing of following formula ,

$$
\begin{equation*}
\frac{\Delta \mathrm{L}}{\mathrm{~L}}=\beta \Delta \mathrm{R} \tag{Ap3.1}
\end{equation*}
$$

where $\beta$ is constant or scaling factor. This formula is the differential equation to generate the exponential function as we see in Section 3.
(Ap3.1) is the same formula with that of the Weber's law, but no relationship with our sensation. Thus, we know that the formula like (Ap3.1) is a basic one that expresses the natural phenomenon which obeys the exponential law. In case of Fig.Ap3-1, the exponential function became

$$
\mathrm{L}=11.56 \exp (0.166 \mathrm{R}) \quad(\mathrm{Ap} 3.2)
$$

where constant 11.56 is a millimeter unit in scale of the photograph and the scaling factor


$$
\beta V=1 \text { or } V_{\beta}=\frac{1}{\beta} \quad(A p 3.7),
$$

where we define $V_{\beta}$ as above if necessary. At $V=V_{\beta}(=1 / \beta), T=T_{\text {min }}$.
The instantaneous flow
$\mathrm{q}=\frac{1}{\mathrm{~T}}$ becomes the maximum $\mathrm{q}_{\max }=\frac{1}{\mathrm{~T}_{\text {min }}}$.

Fig.Ap3-1 The edge slope of Mount Fuji is shown mathematically with the exponential function

## Appendix 3-2

From Fechner's law

$$
\begin{aligned}
& X=X_{0} \exp (\beta V) \quad(\operatorname{Ap} 3.3), \\
& T=\frac{X_{0}}{V} \exp (\beta V) \\
& (\operatorname{Ap~3.4).}
\end{aligned}
$$

Since (Ap3.3) is a concave function with respect to V , the minimum $\mathrm{T}\left(=\mathrm{T}_{\min }\right)$ exixts. Note that $1 / T_{\min }$ gives the instantaneous maximum flow $q_{\text {max }}$. That is, if $T_{\text {min }}$ continued for some time elapsed, the number $\left(\mathrm{N}_{30}\right)$ of flowing car within for example 30 seconds will be

$$
\mathrm{N}_{30}=\frac{30}{\mathrm{~T}_{\min }} \quad(\mathrm{Ap} 3.5) .
$$

$\mathrm{T}_{\text {min }}$ can be derived from the derivative of (Ap 3.4) with respect to V as follows:

$$
\frac{\mathrm{dT}}{\mathrm{dV}}=\frac{\mathrm{X}_{0}}{\mathrm{~V}} \exp \left(-\frac{1}{\mathrm{~V}}+\beta\right) \quad(\operatorname{Ap} 3.6)
$$

From $\frac{d T}{d V}=0$, we get

