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Control of Inverted Pendulum using Nonlinear Fuzzy Servo Control

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1. Introduction

In this paper, the controller design for nonlinear servo system by using fuzzy method will be discussed. Inverted pendulum system has been used for the simulation to illustrate the effectiveness of the proposed method. The inverted pendulum system has been choosing because it is the suitable example to use for investigation and verification of various control methods for dynamic systems. The result shows that the proposed method can stabilize the system.

2. System Description

Let the original system S be a nonlinear system one as

$$\dot{x} = f(x,u) + d \quad (1)$$

$$y = g(x) + d_o \quad (2)$$

where, $x \in R^n$, $u \in R^m$,$y \in R^l$,$d \in R^n$,$d_o \in R^l (m \geqq l)$ with,

$$\delta x = x - x_i \quad (3)$$

$$\delta u = u - u_i. \quad (4)$$

where x is state vector, y is control output, u is control input, d is state disturbance and d_o is output disturbance. n, m and l are dimensions of state, input and output of the system. The system can be linearized by applying Taylor expansion to (1) and (2) around operating point (x_i, u_i), where

$$\delta \dot{x} = \frac{\partial}{\partial x^T} f(x_i, u_i)\,\delta x + \frac{\partial}{\partial u^T} f(x_i, u_i)\,\delta u + f(x_i, u_i) + d \quad (5)$$

$$y = \frac{\partial}{\partial x^T} g(x_i)\,\delta x + g(x_i) + d_o. \quad (6)$$

The linear approximated system S_i can be represented as;

$$\dot{x} = A_i x + B_i u + d_{xi} \quad (7)$$

$$y = C_i x + d_{oi} \quad (8)$$

where,

$A_i = \frac{\partial}{\partial x^T} f(x_i, u_i)\,, B_i = \frac{\partial}{\partial u^T} f(x_i, u_i)\,, C_i =$

$\frac{\partial}{\partial x^T} g(x_i)$, $d_{xi} = f(x_i, u_i) - A_i - B_i u_i + d$, $d_{oi} = g(x_i) - C_i x_i + d_o$.

Let error of the system be as following

$$\dot{v} = y - r. \tag{9}$$

Derive the augemented system from (7)and (9).

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ C_i & 0 \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u + \begin{bmatrix} d_{xi} \\ -r \end{bmatrix} \tag{10}$$

Equation(10) can be re-written as

$$\dot{z} = A_{zi} z + B_{zi} u + d_{zi} \tag{11}$$

where,

$z = \begin{bmatrix} x \\ v \end{bmatrix}$, $A_{zi} = \begin{bmatrix} A_i & 0 \\ C_i & 0 \end{bmatrix}$, $B_{zi} = \begin{bmatrix} B_i \\ 0 \end{bmatrix}$,

$d_{zi} = \begin{bmatrix} d_{xi} \\ d_{oi} - r \end{bmatrix}$.

The system that has ability to follow the reference given is called as servo system.This system in (11) is controllable if controlability condition satisfies.

$$rank\begin{bmatrix} B_{zi} & A_{zi}B_{zi} & A_{zi}^2 B_{zi} & \cdots & A_{zi}^{n+l-1} B_{zi} \end{bmatrix} = n+l. \tag{12}$$

This condition is equivalent to the next one.

$$rank\begin{bmatrix} B_i & A_i B_i & A_i^2 B_i & \cdots & A_i^{n-1} B_i \end{bmatrix} = n$$
$$rank\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} = n+l \tag{13}$$

3. Fuzzy Servo Controller

In this section we apply fuzzy method to develop the control rule for the nonlinear system. First, we divide the operating region of nonlinear system into small area, and treat as a collection of local linear servo systems (Fig.1).

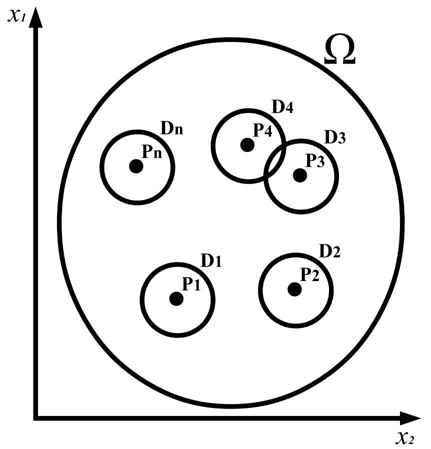

Fig. 1 Partition of driving domain Ω

Fuzzy method has been apply to each local linear system and combines it and can be define as

$$R_i: \quad IF \quad z_n \in D_i \quad THEN \quad S \ is \ S_i. \tag{14}$$

Where, D_i is the local driving area, S is the nonlinear system, S_i is the local linear system and the fuzzy rules are $i = -N, \cdots N$.$z_v \in R^n$ is a vector of nonlinear elements, and

$$z_v = C_v \begin{bmatrix} x \\ u \end{bmatrix}. \tag{15}$$

$z_v \in R^n$ is a vector of nonlinear elements, which are included in controlled system in the equation.$C_v \in R^{n_v \times (n+m)}$ is a matrix of which its elements are 1 or 0. For example, let

$$x^T = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}, \quad u^T = \begin{pmatrix} u_1 & u_2 \end{pmatrix}. \tag{16}$$

In case of x_1, x_2, u_1 are nonlinear, z_v and C_v can be defined as

$$z_v = \begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix}, \quad C_v = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{17}$$

Let ω_i define as Gaussian type fuzzy membership function (Fig.2) as following.

$$\omega_i = e^{-(z_v - z_{vi})^T Q_v (z_v - z_{vi})} \tag{18}$$

$$Q_v = Q_v^T > 0 \tag{19}$$

ρ_i can be define as,

$$\rho_i = \frac{\omega_i}{\sum_{i=-N}^{+N} \omega_i}. \quad (20)$$

The Fuzzy Servo System can be re-write as,

$$\dot{z} = \sum_{i=-N}^{+N} \rho_i \left(A_{zi}x + B_{zi}u + d_{zi}\right). \quad (21)$$

In order to develop new control rule, we calculate each local linear system control rule by using

$$u = -K_i z. \quad (22)$$

K_i is the feedback coefficient matrix of the input. This matrix calculated using Hikita method [3]. Let

$$\begin{aligned} f_{ij} &= -(\lambda_j I - A_{zi})^{-1} B_{zi} g_j \quad (23) \\ (j &= 1, 2, 3, ...n + l), \\ g_j &\in R^m, f_{ij} \in R^{n+l}. \end{aligned}$$

λ_j be the eigenvalues and f_{ij} be eigenvector of the system. The feedback co-efficient matrix can be represented as

$$\begin{aligned} K_i &= \begin{bmatrix} g_{i1} & \dots & g_{i(n+l)} \end{bmatrix} \begin{bmatrix} f_{i1} & \dots & f_{i(n+l)} \end{bmatrix}^{-1} \\ K_i &= \begin{bmatrix} k_{i1} & \dots & k_{i(n+l)} \end{bmatrix}. \quad (24) \end{aligned}$$

Applied fuzzy method to each local linear system and combines it as new control law. The control input u is

$$u = -\sum_{i=-N}^{+N} \rho_i K_i z. \quad (25)$$

The input u in (25) are applied to system in (1) and (2).

4. Simulation and Results

The inverted pendulum used in this simulation is shown in Fig.3. It consists of cart and a pendulum. The cart is free to move the horizontal direction when the force F applies to it. We assume that the mass of the pendulum and cart are homogenously distributed and concentrated in their center of the gravity and the friction of the cart is proportional only to the cart velocity and friction generating by the pivot axis is proportional to the angular velocity of the pendulum. The parameters used for simulation are shown in Table 1.The mathematical model of inverted pendulum system can be described as the following.

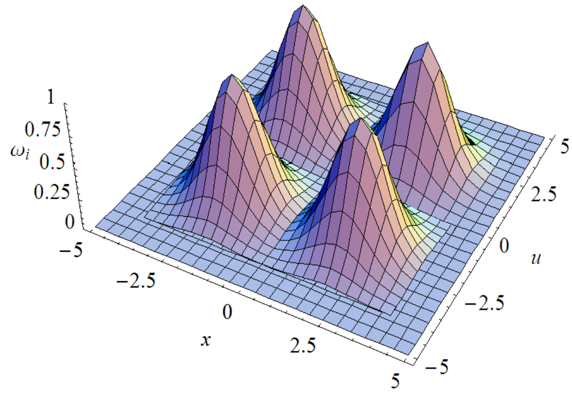

Fig. 2 Gaussian type fuzzy membership function ω_i

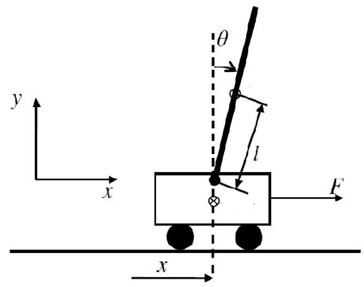

Fig. 3 Inverted Pendulum system

$$(M+m)\ddot{x} + mlcos\theta\ddot{\theta} + D\dot{x} - mlsin\theta\dot{\theta}^2 = F \quad (26)$$

$$mlcos\theta\ddot{x} + \frac{4}{3}ml^2 ddot\theta - mglsin\theta + Cdot\theta = 0 \quad (27)$$

Table 1 Parameters of Inverted Pendulum System.

Parameter	Description	Value
Mass of the cart	M	0.165kg
Mass of the pendulum	m	$0.12kg$
Distance from pivot to center of mass of the pendulum	l	$0.25m$
Gravitational constant	g	$9.80m/s^2$
Co-efficient of friction for pivot	C	$0.01kgm/s$
Co-efficient of friction for cart	D	$4.0kg/s$

With output as

$$y = x_1 \tag{28}$$

Define an error of the system as

$$\dot{v} = e = y - r. \tag{29}$$

Re-arranged the equation in term of $x_1 = x, x_2 = \dot{x}, x_3 = \theta, x_4 = \dot{\theta}$ and $F = u$. The new equation for inverted pendulum are shown in (30), (31),(32) and (33).

$$\dot{x_1} = x_2 \tag{30}$$

$$\dot{x_2} = 3cosx_3\,(mglsinx_3 - Cx_4) - 4l\left(u - Dx_2 + mlsinx_3x_4^2\right) / \left(4l(m+M) + 3mlcosx_3^2\right) \tag{31}$$

$$\dot{x_3} = x_4 \tag{32}$$

$$\dot{x_3} = -3\left((m+M)(mglsinx_3 - Cx_4) - mlcosx_3(u - Dx_2 + mlsinx_3x_4^2)\right) / \left(ml^2\left(-4(m+M) + 3mcosx_3^2\right)\right) \tag{33}$$

From the equation, we know that the nonlinear co-efficient for the system is x_3 and x_4.The linearization has been done to mathematical model of the inverted pendulum system using Taylor expansion as shown in (5) and (6) around of the operating points for nonlinear variables x_3 and x_4. Where,

$$\delta x_3 = x_3 - x_{3i} \tag{34}$$

$$\delta x_4 = x_4 - x_{4i}. \tag{35}$$

The linearization has been done and the linear servo system can be represent as

$$\dot{z} = A_{zi}z + B_{zi}u + d_{zi}. \tag{36}$$

Where,

$$A_{zi} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{23} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & a_{42} & a_{43} & a_{44} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_{zi} = \begin{bmatrix} 0 & b_{21} & 0 & b_{41} & 0 \end{bmatrix}^T$$

$$z = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & v \end{bmatrix}^T$$

and $d_{zi} = \begin{bmatrix} 0 & d_{21} & 0 & d_{41} & 0 \end{bmatrix}^T$ is disturbance vector. Each coefficient can be described as the following

$$a_{22} = (-2D(5m+8m-3mcos[2x_{3i}]+6mx_{3i}\, sin[2x_{3i}])/(4(m+M)-3mcos^2[x_{3i}])^2$$

$$a_{23} = (4ml^2x_{4i}^2cos[x_{3i}(-m+8M+3mcos[2x_{3i}]) + 3mgl(3m-(5m+8M)cos[2x_{3i}] - 3Cx_{4i}(11m+8M+3mcos[2x_{3i}])\, sin[x_{3i}]/(2l(4(m+M)-3mcos^2[x_{3i}])^2$$

$$a_{24} = (3Ccos[x_{3i}] + v8l^2mx_{4i}sin[x_{3i}] / (4l(m+M)-3mlcos^2x_{3i})$$

$$a_{42} = (3Dcos[x_{3i}](5m+8M-3mcos[2x_{3i}]) + x_{3i}(11m+8M+3mcos[2x_{3i}]sin[x_{3i}])) / (2l(4(m+M)-3mcos^2[x_{3i}])^2$$

$$a_{43} = (3mgl(m+M)cos[x_{3i}](-m+8M+ 3mcos[2x_{3i}]+3x_{4i}ml^2x_{4i} (3m-(5m+8M)cos[2x_{3i}] 6C(m+M)sin[2x_{3i}] /(2l(4l(m+M)-3mlcos^2[x_{3i}])^2$$

$$
\begin{aligned}
a_{43} &= (3(C(m+M)+m^2l^2x_{4i}sin[2x_{3i}]) \\
&\quad /(ml^2(-4(m+M)+3mcos^2[x_{3i}]) \\
b_{21} &= (2(5m+8M-3mcos[2x_{3i}]+6mx_{3i} \\
&\quad sin[2x_{3i}]/(4(m+M)-3mcos^2[x_{3i}])^2 \\
b_{41} &= (-6cos[x_{3i}](5m+8M-3mcos[2x_{3i}]) \\
&\quad +2(11m+8M+3mcos[2x_{3i}]sin[x_{3i}])) \\
&\quad /(4l(4(m+M)-3mcos^2[x_{3i}])^2.
\end{aligned}
$$

The controllability of the Servo system of inverted pendulum has been investigated using (12). In this case, the output of system has been setting as shown in (28) because of the controllability problem. The relationship between the output and the state equation can be proved here. Let the linear system as

$$
\begin{aligned}
\dot{x} &= Ax + Bu + d_x &(37)\\
y &= Cx + d_o &(38)
\end{aligned}
$$

and the error of the system be as following

$$\dot{v} = y - r. \quad (39)$$

To construct the servo system let,

$$z=\begin{bmatrix} x \\ v \end{bmatrix}. \quad (40)$$

Therefore, the servo system can be represent as

$$\dot{z}=\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} d_x \\ d_o - r \end{bmatrix} \quad (41)$$

or

$$\dot{z} = A_z z + B_z u + d_z \quad (42)$$

with input as

$$u = -K_z z. \quad (43)$$

Into (42)

$$\dot{z} = (A_z - B_z K_z) + d_z \quad (44)$$

Assume that $d_x = 0$, $d_o = 0$ and $r = 0$. So that when

$$
\begin{aligned}
\lim_{z\to\infty} z &= 0 \\
z(\infty) &= \lim_{z\to\infty} z \\
0 &= (A_z - B_z K_z)^{-1} d_z &(45)
\end{aligned}
$$

Re-arrange the equation.

$$z(\infty) = -(A_z - B_z K_z)^{-1} d_z \quad (46)$$

$z(\infty)=\begin{bmatrix} x_1(\infty) & x_2(\infty) & x_3(\infty) & x_4(\infty) & v(\infty) \end{bmatrix}=$ 0 because $d_z = 0$. Let,

$$Q_v = \begin{bmatrix} q_3 & 0 \\ 0 & q_4 \end{bmatrix} \begin{bmatrix} 5.0 & 0 \\ 0 & 5.0 \end{bmatrix} \quad (47)$$

and the nonlinear variables x_3 and x_4 can be represent as

$$z_v=\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, \quad C_v = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (48)$$

The fuzzy membership function ω_i can be represent as

$$
\begin{aligned}
\omega_i &= e^{-(z_v - z_{vi})^T Q_v (z_v - z_{vi})} \\
&= e^{-q_3(x_3 - x_{3i})^2 - q_4(x_4 - x_{4i})^2} &(49)
\end{aligned}
$$

So that,

$$\rho_i = \frac{\omega_i}{\sum_{i=-N}^{+N} \omega_i}. \quad (50)$$

The set of poles used is $\lambda = [-1.1 - 1.2 - 1.3 - 1.4 - 1.5]$ .The feedback coefficient matrices are calculated using Hikita Method which describe in section before. Where the control input of the system is

$$u = -\sum_{i=-N}^{+N} \rho_i K_i z. \quad (51)$$

with $x_{3i} = h_3 i(-N_3 \leqq i \leqq N_3), x_{4i} = h_4 i(-N_4 \leqq i \leqq N_4), N_3 = 5, N_4 = 5, N = \{(2N_3+1)(2N_4+1) - 1)\}/2 = 60$, increment $h_3 = 0.05, h_4 = 0.05$. The initial condition is $z_0 = [0 \ \ 0 \ \ \theta \ \ 0 \ \ 0]^T$

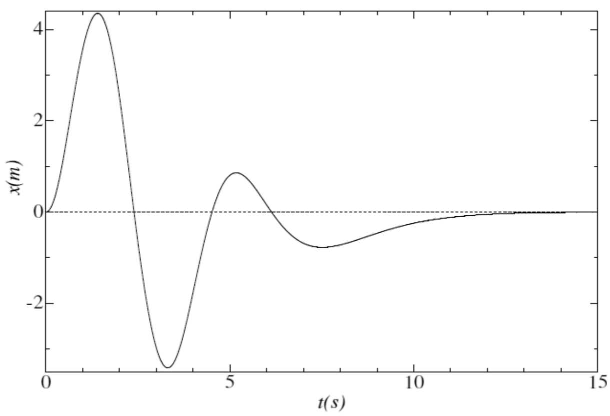

Fig. 4 Output of the simulation $y = x_1 = x$

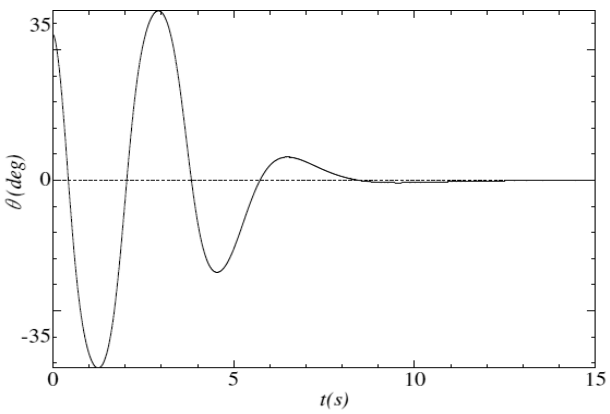

Fig. 5 Output of the simulation $x_3 = \theta$

with value of angle θ is 32^o.The graph in Fig. 5 and Fig.6 shows the output of the simulation. The vertical axes represent displacement of system in unit meter m and radian. The horizontal axis represents time in unit second s. The result shows that not only the output of system $y = x_1 = x$, but the value of $x_3 = \theta$ also converges to zero. The simulation shows that the fuzzy servo control can stabilize inverted pendulum system.

5. Conclusion

In this paper, we apply fuzzy servo control to inverted pendulum system. The implementations of the new control method have been discussed. The simulations have been done and the results shown that method proposed can stabilize the system. As shown in the result, the output of the system follows the reference given and converges to the reference value as desired. For future work, study of characteristic of the proposed method will be done by doing the simulation using other system together with the study of the stability issue.

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